

ON THE KUDLA-RAPOPORT CONJECTURE FOR UNITARY SHIMURA VARIETIES WITH MAXIMAL PARAHORIC LEVEL STRUCTURE AT UNRAMIFIED PRIMES

SUNGYOON CHO, QIAO HE, AND ZHIYU ZHANG

ABSTRACT. In this article, we prove that a version of Tate conjectures for certain Deligne-Lusztig varieties implies the Kudla-Rapoport conjecture for unitary Shimura varieties with maximal parahoric level at unramified primes. Furthermore, we prove that the Kudla-Rapoport conjecture holds unconditionally for several new cases in any dimension.

CONTENTS

1. Introduction	1
2. Rapoport-Zink space and special cycles	9
3. Local density and the modified Kudla-Rapoport conjecture	13
4. Our strategy: $\mathcal{N}_4^{[2]}$	16
5. Horizontal parts of Kudla-Rapoport cycles	19
6. Local modularity and Tate conjectures	23
7. Weighted representation densities and conjectures	28
8. Cho-Yamauchi constants	31
9. Inductive relations among Cho-Yamauchi constants	41
10. Fourier transform	55
11. Tate conjectures and the proof of the main theorem	83
References	92

1. INTRODUCTION

1.1. Introduction.

1.1.1. *Background.* The classical *Siegel-Weil formula* ([Sie35, Sie51, Wei65]) relates special *values* of certain Eisenstein series as theta functions, which are generating series of representation numbers of quadratic forms. Later on, Kudla ([Kud97, Kud04]) proposed an influential program and introduced analogues of theta series in arithmetic geometry. One of the goals of the program is to prove the so-called *arithmetic Siegel-Weil formula* relating the *central derivative* of certain Eisenstein series with a certain arithmetic analogue of theta functions, which is a generating series of arithmetic intersection numbers of n special divisors on Shimura varieties associated to $\mathrm{SO}(n-1, 2)$ or $\mathrm{U}(n-1, 1)$.

Date: March 25, 2024.

For $U(n-1, 1)$ -Shimura varieties, Kudla and Rapoport ([KR11]) formulated a conjectural *local arithmetic Siegel–Weil formula* at an *unramified* place with hyperspecial level, now known as the *Kudla–Rapoport conjecture*. As a local analogue of the arithmetic Siegel–Weil formula, it relates the *central derivative* of local densities of hermitian forms with the arithmetic intersection number of special cycles on unitary Rapoport–Zink spaces. Now this conjectural identity is also known as the Kudla–Rapoport conjecture and was recently proved by Li and Zhang in [LZ22a]. We refer the readers to the introduction of [LZ22a] for more backgrounds and related results.

One of the distinguished features of the hyperspecial case [KR11] is that the corresponding Rapoport–Zink space has good reduction. Accordingly, the analytic side has a clear formulation. A natural and important question is to formulate and prove analogues of the Kudla–Rapoport conjecture when the level structure is non-trivial, where many unexpected new phenomena occur.

At a ramified place, there are two well-studied unitary Rapoport–Zink spaces with different level structures. One of them is the *exotic smooth model* which has good reduction, and the other one is the *Krämer model* which has bad (semistable) reduction. The analogue of Kudla–Rapoport conjecture for the even dimensional exotic smooth model was formulated and proved by Li and Liu in [LL22] using a strategy similar to [LZ22a]. For the Krämer model, corresponding to the bad reduction in this case, the analytic side is more involved. In fact, even the formulation of the conjecture is not clear and needs to be modified. This phenomenon in the presence of bad reduction was first discovered by Kudla and Rapoport in [KR00] via explicit computation in their study of the Drinfeld p -adic half plane. A similar computation was also done in [San17, HSY20] for unitary special cycles on the Krämer models of the Drinfeld p -adic half plane. The Kudla–Rapoport conjecture for Krämer models, in general, was formulated in [HSY23] with conceptual formulation for the modification and proved in [HLSY23].

The present paper focus on Kudla–Rapoport conjectures with maximal parahoric level structures at an unramified prime, where more new phenomena occur. If the level structure is almost self-dual, a Kudla–Rapoport type formula was obtained in [San17] when $n = 2$ by explicit computation and established in general in [LZ22a] by relating it with the hyperspecial case. At an unramified prime with general maximal parahoric level structure, such formulation was first given in [Cho22b] in terms of weighted local density taking advantage of a duality between two Rapoport–Zink spaces (see §7). In this paper, we also give another formulation in the spirit of [HSY23, HLSY23] when the intersection number involved is purely contributed by \mathcal{Z} -cycles.

Assuming a version of Tate conjectures for certain Deligne–Lusztig varieties, the present paper settles this conjecture for any n and any maximal parahoric level structures. We are able to prove the conjecture unconditionally for several special level structures for any n . In particular, we will give a new proof of the almost self-dual case first proved in [LZ22a]. The main results we obtained should be useful to relax the local assumptions in the arithmetic Siegel–Weil formula for $U(n-1, 1)$ -Shimura varieties by allowing maximal parahoric levels at unramified primes. Also, it may be applied to relax the local assumptions in the arithmetic inner product formula in [LL21, LL22] and the p -adic arithmetic inner product formula by Disegni and Liu in [DL22].

1.1.2. *Kudla–Rapoport conjecture.* Let p be an odd prime. Let F_0 be a finite unramified extension of \mathbb{Q}_p with residue field $k = \mathbb{F}_q$. Let F be an unramified quadratic extension of F_0 . Let π be a uniformizer of both F and F_0 . Let \check{F} be the completion of the maximal unramified extension of F . Let $O_F, O_{\check{F}}$ be the ring of integers of F, \check{F} respectively.

Let $n \geq 2$ be an integer. To define the unitary Rapoport–Zink space with maximal parahoric level structure, we fix a supersingular hermitian O_F -modules \mathbb{X} of signature $(1, n-1)$ over \bar{k} . The Rapoport–Zink space $\mathcal{N} = \mathcal{N}_n^{[h]}$ is the formal scheme over $\mathrm{Spf} O_{\check{F}}$ parameterizing hermitian formal O_F -modules X of signature $(1, n-1)$ and type h (see Definition 2.1) within the quasi-isogeny class of \mathbb{X} . The space \mathcal{N} is locally of finite type, and semistable of relative dimension $n-1$ over $\mathrm{Spf} O_{\check{F}}$.

Let $\bar{\mathbb{E}}$ be the framing hermitian O_F -modules of signature $(0, 1)$ over \bar{k} . We define *space of quasi-homomorphisms* to be $\mathbb{V} = \mathbb{V}_n := \mathrm{Hom}_{O_F}(\bar{\mathbb{E}}, \mathbb{X}) \otimes_{O_F} F$. We can associate \mathbb{V} with a natural F/F_0 -hermitian form to make \mathbb{V} a non-degenerate F/F_0 -hermitian space of dimension n . For any subset $L \subset \mathbb{V}$, we define the *special cycle* $\mathcal{Z}(L)$ (resp. $\mathcal{Y}(L)$) (see §2.2) to be the deformation locus of L (resp. $\lambda \circ L$) in $\mathcal{N}_n^{[h]}$.

Given an O_F -lattice $L \subset \mathbb{V}$ of full rank n , we can define two integers: the *arithmetic intersection number* $\mathrm{Int}(L)$ and the *modified derived local density* $\partial \mathrm{Den}(L)$.

Definition 1.1. Let $L \subset \mathbb{V}$ be an O_F -lattice and x_1, \dots, x_n be an O_F -basis of L . We define the *arithmetic intersection number*

$$(1.1) \quad \mathrm{Int}_{n,h}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) \in \mathbb{Z},$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the special divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on \mathcal{N} , and χ denotes the Euler–Poincaré characteristic. By Proposition 2.12, $\mathrm{Int}_{n,h}(L)$ is independent of the choice of the basis x_1, \dots, x_n and hence is a well-defined invariant of L itself.

To define the *modified derived local density* $\partial \mathrm{Den}(L)$, we need to introduce local densities first. Let M be another hermitian O_F -lattice (of arbitrary rank) and $\mathrm{Herm}_{L,M}$ denote the O_{F_0} -scheme of hermitian O_F -module homomorphisms from L to M . Then we define the corresponding *local density* to be

$$\mathrm{Den}(M, L) := \lim_{d \rightarrow +\infty} \frac{|\mathrm{Herm}_{L,M}(O_{F_0}/\pi^d)|}{q^{d \cdot d_{L,M}}},$$

where $d_{L,M}$ is the dimension of $\mathrm{Herm}_{L,M} \otimes_{O_{F_0}} F_0$. Let I_k be an unimodular hermitian O_F -lattice of rank k . It is well-known that there exists a *local density polynomial* $\mathrm{Den}(M, L, X) \in \mathbb{Q}[X]$ such that for any integer $k \geq 0$,

$$(1.2) \quad \mathrm{Den}(M, L, (-q)^{-k}) = \mathrm{Den}(I_k \oplus M, L).$$

Here $I_k \oplus M$ denotes the orthogonal direct sum of I_k and M .

When M also has rank n and $M \otimes_{O_F} F$ is not isometric to $L \otimes_{O_F} F$, we have $\mathrm{Den}(M, L) = 0$. In this case we write

$$\mathrm{Den}'(M, L) := - \frac{d}{dX} \Big|_{X=1} \mathrm{Den}(M, L, X),$$

and define the (normalized) *derived local density*

$$(1.3) \quad \text{Den}'_{n,h}(L) := \frac{\text{Den}'(I_{n,h}, L)}{\text{Den}(I_{n,h}, I_{n,h})} \in \mathbb{Q}.$$

Here $I_{n,h}$ is a hermitian lattice with moment matrix $\text{Diag}((1)^{n-h}, (\pi)^h)$. When the n and h are clear in the context, we also simply denote it as $\text{Den}'(L)$.

Then the naive analogue of the Kudla–Rapoport conjecture for $\mathcal{N}_n^{[h]}$ should be the identity

$$\text{Int}_{n,h}(L) \stackrel{?}{=} \text{Den}'_{n,h}(L).$$

However, as we mentioned before, since $\mathcal{N}_n^{[h]}$ has bad reduction, some modification for the analytic side is needed. Indeed, a similar consideration as in [HSY23, HLSY23] shows that naive analogue cannot be true for trivial reasons. For example, if $h > 0$, then $\text{Int}(L)$ vanishes by definition if L has I_{n-h+1} as a direct summand while $\text{Den}'(L)$ does not vanish for such L by some direct computation. We call an integral O_F -lattice $\Lambda \subset \mathbb{V}$ a *vertex lattice of type t* if Λ^\vee/Λ is a k -vector space of dimension t . In particular, for a vertex lattice $\Lambda \subset \mathbb{V}$ of type $t < h$, we have $\text{Int}(\Lambda) = 0$, while $\text{Den}'(\Lambda) \neq 0$ in general.

In order to have $\text{Int}(\Lambda_t) = \partial\text{Den}(\Lambda_t)$ for vertex lattice Λ_t of type $t < h$, we define $\partial\text{Den}(L)$ by modifying $\text{Den}'(L)$ with a linear combination of the (normalized) *local densities*

$$(1.4) \quad \text{Den}_{n,t}(L) := \frac{\text{Den}(\Lambda_t, L)}{\text{Den}(\Lambda_t, \Lambda_t)} \in \mathbb{Z}.$$

In fact $\mathbb{V} \not\cong I_n^{[h]} \otimes_{O_F} F$. As a result, if $\Lambda_t \subset \mathbb{V}$, then t and h have different parity.

Definition 1.2 (Definition 3.2). Let $L \subset \mathbb{V}$ be an O_F -lattice. Define the *modified derived local density*

$$(1.5) \quad \partial\text{Den}_{n,h}(L) := \text{Den}'_{n,h}(L) + \sum_{i=0}^{\lfloor \frac{h-1}{2} \rfloor} c_{n,h-1-2i} \cdot \text{Den}_{n,h-1-2i}(L).$$

The coefficients $c_{n,i} \in \mathbb{Q}$ here are chosen to satisfy

$$(1.6) \quad \partial\text{Den}_{n,h}(\Lambda_i) = 0, \quad \text{for } 0 \leq i \leq h-1 \text{ and } i \equiv h-1 \pmod{2},$$

which turns out to be a linear system in $(c_{n,i})$ with a unique solution.

Finally, we propose the following Kudla–Rapoport conjecture for $\mathcal{N}_n^{[h]}$.

Conjecture 1.3 (Conjecture 3.3, Conjecture 7.7, Conjecture 8.8). *Let $L \subset \mathbb{V}$ be an O_F -lattice. Then we have*

$$\text{Int}_{n,h}(L) = \partial\text{Den}_{n,h}(L).$$

Remark 1.4. We remark that $\text{Den}'_{n,h}(L)$ is not an integer in general. Only the modified $\partial\text{Den}_{n,h}(L)$ is an integer. However, a priori, this is not clear at all.

The main purpose of this paper is to prove Conjecture 1.3 assuming a version of Tate conjectures for certain Deligne–Lusztig varieties.

Theorem 1.5 (Theorem 11.4). *Let $L \subset \mathbb{V}$ be an O_F -lattice of rank n . Assuming Conjecture 6.3, we have*

$$\text{Int}_{n,h}(L) = \partial\text{Den}_{n,h}(L).$$

In this paper, we also verified Conjecture 6.3 for $\mathcal{N}_n^{[1]}$, $\mathcal{N}_n^{[n-1]}$ and $\mathcal{N}_4^{[2]}$ unconditionally.

Theorem 1.6 (Theorem 6.5, Theorem 11.5). *Assume (n, h) is one of the following cases: $(n, 1)$, $(n, n-1)$ and $(4, 2)$. Then Conjecture 1.3 holds unconditionally.*

Remark 1.7. Conjecture 1.3 for $\mathcal{N}_n^{[0]}$ was the original Kudla-Rapoport conjecture proposed in [KR11] and proved in [LZ22a]. The case $(n, h) = (n, 1)$ was first proved by [LZ22a] using certain Hecke correspondence that relates $\mathcal{N}_n^{[1]}$ with \mathcal{N}_{n+1} . Our proof is an attempt to prove the general case uniformly in a way closer to the proof for $\mathcal{N}_n^{[0]}$ in [LZ22a].

Remark 1.8. Although $\mathcal{N}_n^{[1]} \cong \mathcal{N}_n^{[n-1]}$ by a natural duality, \mathcal{Z} -cycles on $\mathcal{N}_n^{[1]}$ will be transformed into \mathcal{Y} -cycles on $\mathcal{N}_n^{[n-1]}$. Hence, Theorem 1.6 for $\mathcal{N}_n^{[n-1]}$ is different with the case for $\mathcal{N}_n^{[1]}$. In particular, Theorem 1.6 for $\mathcal{N}_n^{[n-1]}$ and $\mathcal{N}_4^{[2]}$ are new.

We remark that Conjecture 1.3 is based on a different viewpoint from the one formulated in [Cho22b]. The conjecture formulated in [Cho22b] is inspired by the duality $\mathcal{N}_n^{[h]} \cong \mathcal{N}_n^{[n-h]}$ and is more general in the sense that it also considers the case when the intersection is between \mathcal{Z} -cycles and \mathcal{Y} -cycles. However, since our main theorem in this paper is about intersections between \mathcal{Z} -cycles and the above formulation is closer to previously studied cases e.g. [LZ22a],[LL22],[LZ22b],[HLSY23], we state this new formulation in the introduction. We refer the reader to Conjecture 7.6 for the general conjecture formulated in [Cho22b] and Conjecture 7.7 for the specialized conjecture about the intersection between \mathcal{Z} -cycles. We show that conjectures 1.3 and 7.7 are in fact equivalent in Proposition 8.9, which is interestingly a nontrivial fact to prove.

1.1.3. *Strategy and novelty.* Our general strategy is similar to the strategy of [LZ22a] using the local modularity and uncertainty principle. More precisely, fix an O_F -lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$ and consider functions on $\mathbb{V} \setminus L_F^\flat$,

$$\text{Int}_{L^\flat}(x) := \text{Int}(L^\flat + \langle x \rangle), \quad \partial\text{Den}_{L^\flat}(x) := \partial\text{Den}(L^\flat + \langle x \rangle).$$

We need to show $\text{Int}_{L^\flat} = \partial\text{Den}_{L^\flat}$ as functions on $\mathbb{V} \setminus L_F^\flat$. First, we have a natural decomposition of Int_{L^\flat} and $\partial\text{Den}_{L^\flat}$ into horizontal parts and vertical parts:

$$\text{Int}_{L^\flat} = \text{Int}_{L^\flat, \mathcal{H}} + \text{Int}_{L^\flat, \mathcal{V}}, \quad \partial\text{Den}_{L^\flat} = \partial\text{Den}_{L^\flat, \mathcal{H}} + \partial\text{Den}_{L^\flat, \mathcal{V}}.$$

For the horizontal parts, we have $\text{Int}_{L^\flat, \mathcal{H}} = \partial\text{Den}_{L^\flat, \mathcal{H}}$ by direct comparison. For the vertical parts, $\text{Int}_{L^\flat, \mathcal{V}}$ and $\partial\text{Den}_{L^\flat, \mathcal{V}}$ satisfy “local modularity” in the sense that their Fourier-transforms have nice behaviors.

Due to the existence of nontrivial levels, new phenomenons have already shown up for the horizontal part. We can decompose the horizontal cycles into primitive horizontal cycles indexed by “horizontal lattices” and essentially reduce the computation to $n=2$, similar to [LZ22a]. However, first of all, we have two different types of the “horizontal lattices” indexing the primitive horizontal cycle. Moreover, the primitive piece indexed by one of them admits a further decomposition into

mixed special cycles. Here mixed special cycles mean that they are intersections between both \mathcal{Z} -cycles and \mathcal{Y} -cycles. We refer the readers to Theorem 5.3 for more details.

Now we turn our attention to the vertical part. We discuss the geometric side first. For a curve $C \subset \mathcal{N}_n^{[h]}$, let $\text{Int}_C(\mathcal{Z}(x)) := \chi(\mathcal{N}_n^{[h]}, \mathcal{O}_C \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)})$ and $\text{Int}_C(\mathcal{Y}(x)) := \chi(\mathcal{N}_n^{[h]}, \mathcal{O}_C \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Y}(x)})$. If C lies in $\mathcal{N}_2^{[1]}$ or $\mathcal{N}_3^{[0]}$ and is embedded into $\mathcal{N}_n^{[h]}$ via an embedding of $\mathcal{N}_2^{[1]}$ or $\mathcal{N}_3^{[0]}$ into $\mathcal{N}_n^{[h]}$, then explicit computation shows that

$$(1.7) \quad \widehat{\text{Int}}_C(\mathcal{Z}(x)) = -q^{-h} \cdot \text{Int}_C(\mathcal{Y}(x)),$$

where $\widehat{\text{Int}}_C(\mathcal{Z}(x))$ denotes the Fourier transform of $\text{Int}_C(\mathcal{Z}(x))$ (see §1.2 for more details). We call such a curve C a special curve. This was first observed in [LZ22a] when $h = 0$. In fact, when $h = 0$, the more general identity

$$\widehat{\text{Int}}_{L^b, \mathcal{Y}}(\mathcal{Z}(x)) = -\text{Int}_{L^b, \mathcal{Y}}(\mathcal{Y}(x))$$

was proved in [LZ22a], which we call *local modularity* in this case. When $h \geq 0$ and C is special, (1.7) was observed in [Zha22] based on computation of intersection numbers on $\mathcal{N}_2^{[1]}$ established in [San17]. In [Zha22], this local modularity was used to prove arithmetic transfer conjecture in a similar setting.

Nevertheless, it is by no means clear that this will be true in general. One observation is that if $\text{Int}_{L^b, \mathcal{Y}}(x)$ can be written as a linear combination of $\text{Int}_C(x)$ for special C , then we have

$$(1.8) \quad \widehat{\text{Int}}_{L^b, \mathcal{Y}}(x) = \begin{cases} -q^{-h} \cdot \text{Int}_{n-1, h-1, \mathcal{Y}}(L^b) & \text{if } \text{val}(x) = -1, \\ 0 & \text{if } \text{val}(x) \leq -2. \end{cases}$$

Although it is not clear whether $\text{Int}_{L^b, \mathcal{Y}}(x)$ can be written as a linear combination of $\text{Int}_C(x)$ for special C , one can always try to test the corresponding speculation on the analytic side, which is a central idea for many of the past works. Indeed, for example, we recognized that there should exist two different types of horizontal lattices by testing the analytic side first (the horizontal part goes to infinity when $\text{val}(x)$ goes to infinity). Also, it is really the computation of $\widehat{\partial \text{Den}}_{L^b, \mathcal{Y}}(x)$ in the $\mathcal{N}_4^{[2]}$ case inspires us what we should expect in general for the geometric side as we explained in §4.

On the other hand, once we have a big picture for the geometric side with the help of explicit computation from the analytic side, the insight from the geometric side also serves as a guiding principle to obtain purely analytic results. For example, [HLSY23, Proposition 7.5] is inspired by the fact that $\mathcal{Z}(L)$ is empty for non-integral L , and the proof of [HLSY23, Proposition 7.5] gives important hints for how to prove the rest major analytic results of [HLSY23]. This is also what happened in the current case.

Indeed, as one of the main results, we manage to prove an analytic analogue of (1.8) unconditionally. More precisely, we prove $\partial \text{Den}_{L^b, \mathcal{Y}}(x)$ can be extended to \mathbb{V} as a locally constant function and

$$(1.9) \quad \widehat{\partial \text{Den}}_{L^b, \mathcal{Y}}(x) = \begin{cases} -q^{-h} \cdot \text{Int}_{n-1, h-1, \mathcal{Y}}(L^b) & \text{if } \text{val}(x) = -1, \\ 0 & \text{if } \text{val}(x) \leq -2. \end{cases}$$

As a result, assuming (1.8), for any x with $\text{val}(x) < 0$, we have

$$\widehat{\text{Int}}_{L^b}(x) - \widehat{\partial\text{Den}}_{L^b}(x) = \widehat{\text{Int}}_{L^b, \gamma}(x) - \widehat{\partial\text{Den}}_{L^b, \gamma}(x) = 0.$$

Then the identity $\text{Int}_{L^b}(x) - \partial\text{Den}_{L^b}(x) = 0$ follows from the uncertainty principle as in [LZ22a].

We remark that (1.9) suggests that $\text{Int}_{L^b, \gamma}(x)$ might be written as a linear combination of $\text{Int}_C(x)$ for special curve C . When $h = 0$, this is indeed the case and was proved in [LZ22a, Corollary 5.3.3]. Therefore we propose this statement as a conjecture in Conjecture 6.3.

In order to establish (1.9), we make use of the *primitive decomposition* of the local density polynomial into primitive local density polynomials and obtain a decomposition of $\partial\text{Den}(L)$:

$$(1.10) \quad \partial\text{Den}(L) = \sum_{L \subset L'} \partial\text{Pden}(L'),$$

where L' runs over O_F -lattices in L_F containing L , and the symbol Pden stands for the primitive version of Den (see (3.6)). The key reason to consider this primitive decomposition and $\partial\text{Pden}(L)$ is that we usually have a very simple formula for $\partial\text{Pden}(L)$ which eventually makes the computation about $\partial\text{Den}(L)$ possible. This is the case for all the previously proved Kudla-Rapoport type formulas, e.g. [LZ22a], [LL22], [LZ22b], and [HLSY23].

Although a similar approach to compute $\partial\text{Pden}(L)$ as in [HLSY23] may be directly generalized to the current situation, we decide to generalize the main result of [Cho23] to obtain an explicit formula of $\partial\text{Pden}(L)$ (see Proposition 8.17). One of the reasons we choose this approach is due to the fact that Proposition 8.17 itself is already a certain induction formula that reduces the computation of $\partial\text{Pden}(L)$ to the good reduction case (see Remark 8.18), which is particularly handy in order to prove Theorem 1.9 below.

One of the major new phenomenons and difficulties we found and overcame in this paper is the fact that when there is a non-trivial level structure, $\partial\text{Pden}(L)$ *no longer* has a simple formula in general. This can be seen via some explicit computation using Proposition 8.17. Nevertheless, inspired by an attempt to compute the Fourier transform of $\partial\text{Den}_{L^b, \gamma}(x)$ (e.g. to obtain (1.9)), we find simple inductive formulas for $\partial\text{Pden}(L)$ which suffices to control the Fourier transform of $\partial\text{Den}_{L^b, \gamma}(x)$. In fact, such inductive formulas also hold in all the previously studied cases and the simple formulas for $\partial\text{Pden}(L)$ in these cases follow as a direct corollary.

More precisely, let $t_i(L)$ and $t_{\geq i}(L)$ be the number of the fundamental invariants of L that is exactly i and at least i respectively. Then we have the following.

Theorem 1.9 (Theorem 9.4). *Let L be a hermitian lattice with $\text{val}(L) \not\equiv h \pmod{2}$. We have $\partial\text{Pden}_{n,h}(L)$ depends only on $(t_{\geq 2}(L), t_1(L), t_0(L))$. For simplicity, we denote it as $D_{n,h}(t_2, t_1, t_0)$. Moreover, assume that $(t_2 - 1, t_1 + 1, t_0) \neq (n - h, h, 0)$, $t_0 \leq n - h$ and $t_2 \geq 1$, then we have*

$$D_{n,h}(t_2, t_1, t_0) - D_{n,h}(t_2 - 1, t_1 + 1, t_0) = -(-q)^{2n-h-1-t_1-2t_0} D_{n-1,h-1}(t_2 - 1, t_1, t_0).$$

The above inductive formula is complemented by the following simple formulas for special $D_{n,h}(t_2, t_1, t_0)$.

Theorem 1.10. *With the same notations and assumptions as in Theorem 1.9, we have the following.*

(1) (Lemma 9.5, Proposition 9.6) If $t_0 > n - h$, then $D_{n,h}(t_2, t_1, t_0) = 0$.

(2) (Theorem 9.8 (1)) If $t_2 = 0$ and $h + 1 \leq t_1$. Then, we have

$$D_{n,h}(0, t_1, t_0) = \frac{\prod_{l=h+1}^{t_1} (1 - (-q)^l)}{(1 - (-q)^{t_1-h})}.$$

(3) (Theorem 9.8 (2)) If $t_2 = 1$ and $h - 1 \leq t_1$. Then, we have

$$D_{n,h}(1, t_1, t_0) = \begin{cases} 1 & \text{if } t_1 = h - 1, h; \\ \prod_{l=h+1}^{t_1} (1 - (-q)^l) & \text{if } t_1 \geq h + 1. \end{cases}$$

Theorems 1.9 and 1.10 will be proved in §9 using the formula for $\partial\text{Pden}(L)$ obtained in [Cho23]. As we have remarked, the method developed in [HLSY23] may also be adapted to the current case. However, we found that using the formulas in [Cho23] is easier for our purpose so we stick with this approach. Note that the formulas in [Cho23] are derived via a duality between weighted local densities in analogy with the duality between $\mathcal{N}_n^{[h]}$ and $\mathcal{N}_n^{[n-h]}$, and this might be a particular reason for why this formula so applicable in the current case.

With the help of formulas for $\partial\text{Pden}(L)$, we finally prove (1.9) via certain involved weighted lattice counting in §10 via a similar method as in [LZ22b, HLSY23]. However, since $\partial\text{Pden}(L)$ now depends on $t_{\geq 2}(L), t_1(L)$ and $t_0(L)$, the counting becomes much more involved compared with the ones in [LZ22b, HLSY23].

1.2. Notation and terminology.

- Let p be an odd prime. Let F_0 be a finite unramified extension of \mathbb{Q}_p with residue field $k = \mathbb{F}_q$. Let F be the unramified quadratic extension of F_0 . Let π be a uniformizer of F and F_0 . Let \check{F} be the completion of the maximal unramified extension of F . Let $O_F, O_{\check{F}}$ be the ring of integers of F, \check{F} respectively.
- We say a sublattice of a hermitian space is non-degenerate if the restriction of the hermitian form to it is non-degenerate.
- In this paper, without explicit mentioning a lattice means a non-degenerate hermitian O_F -lattice. Unless otherwise stated, the symbol L always means a non-degenerate lattice of rank n with a hermitian form $(,)$.
- We define L^\vee to be the dual lattice of L with respect to the hermitian form $(,)$. If $L \subset L^\vee$, we say L is integral. If $L \subset L^\vee \subset \pi^{-1}L$, we say L is a vertex lattice.
- We say that a basis $\{\ell_1, \dots, \ell_n\}$ of L is a normal basis (which always exists) if its moment matrix $T = ((\ell_i, \ell_j))_{1 \leq i, j \leq n}$ is

$$\text{Diag}(\pi^{\alpha_1}, \dots, \pi^{\alpha_n})$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$. Moreover, we define its fundamental invariants (a_1, \dots, a_n) to be the unique nondecreasing rearrangement of $(\alpha_1, \dots, \alpha_n)$.

- We define the valuation of L to be $\text{val}(L) := \sum_{i=1}^n a_i$, where (a_1, \dots, a_n) are the fundamental invariants of L . For $x \in L$, we define $\text{val}(x) = \text{val}((x, x))$, where $\text{val}(\pi) = 1$.
- We call a sublattice $N \subset M$ primitive in M if $\dim_{\mathbb{F}_q} \bar{N} = r(N)$, where $\bar{N} = (N + \pi M)/\pi M$. We also use \bar{L} to denote $L \otimes_{O_F} O_F/(\pi)$.

- We use I_m to denote a unimodular lattice of rank m , and $I_{n,h}$ to denote a hermitian lattice with moment matrix $\text{Diag}((1)^{n-h}, (\pi)^h)$.
- For a hermitian space \mathbb{V} , we let $\mathbb{V}^{?i} := \{x \in \mathbb{V} \mid \text{val}(x) \geq i\}$ where $?$ can be \geq, \leq or $=$.
- Fix an unramified additive character $\psi : F_0 \rightarrow \mathbb{C}^\times$. Here "unramifiedness" means that the conductor of ψ (i.e., the largest fractional ideal in F_0 on which ψ is trivial) is O_{F_0} . For an integrable function f on \mathbb{V} , we define its Fourier transform \widehat{f} to be

$$\widehat{f}(x) := \int_{\mathbb{V}} f(y) \psi(\text{tr}_{F/F_0}(x, y)) dy, \quad x \in \mathbb{V}.$$

We normalize the Haar measure on \mathbb{V} to be self-dual, so $\widehat{f}(x) = f(-x)$. For an O_F -lattice $L \subset \mathbb{V}$ of rank n , we have (under the assumption that F/F_0 is unramified)

$$\widehat{\mathbf{1}}_L = \text{vol}(L) \mathbf{1}_{L^\vee}, \quad \text{and} \quad \text{vol}(L) = [L^\vee : L]^{-1/2} = q^{-\text{val}(L)}.$$

Note that $\text{val}(L)$ can be defined for any lattice L (not necessarily integral) so that the above equality for $\text{vol}(L)$ holds.

1.3. Acknowledgement. We would like to thank Sungmun Cho, Benjamin Howard, Chao Li, Yifeng Liu, Michael Rapoport, Tonghai Yang and Wei Zhang for helpful discussions or comments. Part of the work was done while Q.H. and Z.Z. were participating in the "Algebraic Cycles, L -values, and Euler Systems" program held in MSRI during Spring 2023. We would like to thank MSRI for the excellent work conditions, financial support, and hospitality. S.C. was supported by POSTECH Basic Science Research Institute under the NRF grant number NRF2021R1A6A1A1004294412. Q.H. was partially supported by Columbia University's Faculty Research Allocation Program.

2. RAPOPORT-ZINK SPACE AND SPECIAL CYCLES

2.1. Rapoport-Zink space. In this section, we review the definition and basic properties of Rapoport-Zink space and special cycles.

Definition 2.1. For any Spf $O_{\mathbb{F}}$ -scheme S , a *hermitian formal O_F -module* (X, ι, λ) of signature $(1, n-1)$ and type h over S is the following data:

- X is a strict formal O_{F_0} -module over S of relative height $2n$ and dimension n . Strictness means the induced action of O_{F_0} on $\text{Lie } X$ is via the structure morphism $O_{F_0} \rightarrow \mathcal{O}_S$.
- $\iota : O_F \rightarrow \text{End}(X)$ is an action of O_F on X that extends the action of O_{F_0} . We require that the *Kottwitz condition* of signature $(1, n-1)$ holds for all $a \in O_F$:

$$(2.1) \quad \text{char}(\iota(a) \mid \text{Lie } X) = (T - a)(T - \bar{a})^{n-1} \in \mathcal{O}_S[T].$$

- λ is a polarization on X , which is O_F/O_{F_0} semi-linear in the sense that the Rosati involution Ros_λ induces the non-trivial involution on $\iota : O_F \rightarrow \text{End}(X)$.
- We require that the finite flat group scheme $\text{Ker } \lambda$ over S lies in $X[\pi]$ and is of order q^{2h} .

An isomorphism $(X_1, \iota_1, \lambda_1) \xrightarrow{\sim} (X_2, \iota_2, \lambda_2)$ between two such triples is an O_F -linear isomorphism $\varphi : X_1 \xrightarrow{\sim} X_2$ such that $\varphi^*(\lambda_2) = \lambda_1$. Up to O_F -linear quasi-isogeny compatible with the polarization, there exists a unique such triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over \mathbb{F} . Fix one choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as the *framing object*.

Definition 2.2. Let (Nilp) be the category of O_F -schemes S such that π is locally nilpotent on S . Then the Rapoport–Zink space associated with $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is the functor

$$\mathcal{N}_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})}^{[h]} = \mathcal{N}_n^{[h]} \rightarrow \text{Spf } O_{\check{F}}$$

sending $S \in (\text{Nilp})$ to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$, where

- (X, ι, λ) is a hermitian formal O_F -module of dimension n and type h over S ;
- $\rho : X \times_S \bar{S} \rightarrow \mathbb{X} \times_{\mathbb{F}} \bar{S}$ is an O_F -linear quasi-isogeny of height 0 over the reduction $\bar{S} := S \times_{O_{\check{F}_0}} \mathbb{F}$ such that $\rho^*(\lambda_{\mathbb{X}, \bar{S}}) = \lambda_{\bar{S}}$.

The functor $\mathcal{N}_n^{[h]}$ is representable by a formal scheme over $\text{Spf } O_{\check{F}}$ which is locally formally of finite type by [RZ96]. Moreover, this formal scheme is regular (see [Cho18, Proposition 3.33]). We often simply denote $\mathcal{N}_n^{[h]}$ as \mathcal{N} if the signature and type are clear in the context.

There is an isomorphism $\theta : \mathcal{N}_n^{[h]} \xrightarrow{\sim} \mathcal{N}_n^{[n-h]}$ constructed as follows ([Cho18, Remark 5.2]). For each $S \in (\text{Nilp})$,

$$\begin{aligned} \mathcal{N}_n^{[h]}(S) &\xrightarrow{\theta} \mathcal{N}_n^{[n-h]}(S), \\ (X, i_X, \lambda_X, \rho_X) &\mapsto (X^\vee, \bar{i}_X^\vee, \lambda'_X, (\rho_X^\vee)^{-1}), \end{aligned}$$

where $\lambda'_X : X^\vee \rightarrow X$ is the unique polarization such that $\lambda'_X \circ \lambda_X = i_X(\pi)$, and for $a \in O_F$, the action \bar{i}_X^\vee is defined as $\bar{i}_X^\vee(a) := i_X(a^*)^\vee$. When we denote $\mathcal{N}_n^{[h]}$ as \mathcal{N} , we use \mathcal{N}^\vee to denote $\mathcal{N}_n^{[n-h]}$.

2.2. Special cycles. To define special cycles, we need to fix a hermitian formal O_F -module $(\bar{\mathbb{E}}, i_{\bar{\mathbb{E}}}, \lambda_{\bar{\mathbb{E}}})$ of signature $(0, 1)$ and type 0 over \mathbb{F} . Then we can similarly define a Rapoport–Zink Space $\mathcal{N}_{(\bar{\mathbb{E}}, i_{\bar{\mathbb{E}}}, \lambda_{\bar{\mathbb{E}}})}$ which we denote as \mathcal{N}^0 for simplicity. Recall that there is a unique lifting $(\bar{\mathcal{E}}, \iota_{\bar{\mathcal{E}}}, \lambda_{\bar{\mathcal{E}}})$ of $(\bar{\mathbb{E}}, i_{\bar{\mathbb{E}}}, \lambda_{\bar{\mathbb{E}}})$ over $O_{\check{F}}$.

Definition 2.3. We define the space of special homomorphism to be the F -vector space

$$\mathbb{V} := \text{Hom}_{O_F}(\bar{\mathbb{E}}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We can associate \mathbb{V} with a naturally defined hermitian form as follows. For $x, y \in \mathbb{V}$, we define a hermitian form h on \mathbb{V} as

$$(x, y) = \lambda_{\bar{\mathbb{E}}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}} \circ x \in \text{End}_{O_F}(\bar{\mathbb{E}}) \otimes \mathbb{Q} \xrightarrow{\cong} F.$$

We often omit $i_{\bar{\mathbb{E}}}^{-1}$ via the identification $\text{End}_{O_F}(\bar{\mathbb{E}}) \otimes \mathbb{Q} \simeq F$.

Definition 2.4. [KR11, Definition 3.2], [Cho18, Definition 5.4]

- (1) For $x \in \mathbb{V}$, we define the special cycle $\mathcal{Z}(x)$ to be the closed formal subscheme of $\mathcal{N}^0 \times \mathcal{N}$ with the following property: For each O_F -scheme S such that π is locally nilpotent, $\mathcal{Z}(x)(S)$ is the set of all points $\xi = (\bar{\mathcal{E}}_S, \iota_{\bar{\mathcal{E}}_S}, \lambda_{\bar{\mathcal{E}}_S}, X, i_X, \lambda_X, \rho_X)$ in $\mathcal{N}^0 \times \mathcal{N}(S)$ such that the quasi-homomorphism

$$\rho_X^{-1} \circ x \circ \rho_{\bar{\mathcal{E}}_S} : \bar{\mathcal{E}}_S \times_S \bar{S} \rightarrow X \times_S \bar{S}$$

extends to a homomorphism from $\bar{\mathcal{E}}_S$ to X .

- (2) For $y \in \mathbb{V}$, we define the special cycle $\mathcal{Y}(y)$ in $\mathcal{N}^0 \times \mathcal{N}$ as follows. First, consider the special cycle $\mathcal{Z}(\lambda_{\mathbb{X}} \circ y)$ in $\mathcal{N}^0 \times \mathcal{N}^{\vee}$. This is the closed formal subscheme of $\mathcal{N}^0 \times \mathcal{N}^{\vee}$. We define $\mathcal{Y}(y)$ as $(id \times \theta^{-1})(\mathcal{Z}(\lambda_{\mathbb{X}} \circ y))$ in $\mathcal{N}^0 \times \mathcal{N}$

Note that \mathcal{N}^0 can be identified with $\mathrm{Spf} O_{\check{F}}$, and hence $\mathcal{Z}(x), \mathcal{Y}(y)$ can be identified with closed formal subschemes of \mathcal{N} .

The same proof of [KR14, Proposition 5.9] gives us the following.

Proposition 2.5. [KR14, Proposition 5.9] *The functors $\mathcal{Z}(x)$ and $\mathcal{Y}(y)$ are representable by Cartier divisors of \mathcal{N} .*

Special cycles have the following properties.

Proposition 2.6. [Cho18, Proposition 5.10] *Let $x, y \in \mathbb{V}$.*

- (1) *If $\mathrm{val}((x, x)) = 0$, then $\mathcal{Z}(x) \simeq \mathcal{N}_{n-1}^{[h]}$.*
(2) *If $\mathrm{val}((y, y)) = -1$, then $\mathcal{Y}(y) \simeq \mathcal{N}_{n-1}^{[h-1]}$.*

Proposition 2.7. [Cho18, Proposition 5.11] *Assume that $\mathrm{val}((x, x)) = 0$ and $\mathrm{val}((y, y)) = -1$. Assume further that by rescaling as in the proof of [Cho18, Proposition 5.10], $x^* \circ x = 1$, and $(\lambda_{\mathbb{X}} \circ y)^* \circ (\lambda_{\mathbb{X}} \circ y) = 1$. Here, x^* (resp. $(\lambda_{\mathbb{X}} \circ y)^*$) is the adjoint of x (resp. $(\lambda_{\mathbb{X}} \circ y)$) with respect to the polarizations $\lambda_{\mathbb{X}}$ and $\lambda_{\mathbb{E}}$ (resp. $\lambda'_{\mathbb{X}}$ and $\lambda'_{\mathbb{E}}$). We define $e_x := x \circ x^*$ and $e_y := (\lambda_{\mathbb{X}} \circ y) \circ (\lambda_{\mathbb{X}} \circ y)^*$. Fix isomorphisms*

$$\begin{aligned}\Phi : \mathcal{Z}(x) &\simeq \mathcal{N}_{n-1}^{[h]}, \\ \Psi : \mathcal{Y}(y) &\simeq \mathcal{N}_{n-1}^{[h-1]},\end{aligned}$$

as in Proposition 2.6. Then the following statements hold.

- (1) *For $z \in \mathbb{V}$ such that $(x, z) = 0$, let $z' := (1 - e_x) \circ z$. Then, we have $\Phi(\mathcal{Z}(x) \cap \mathcal{Z}(z)) = \mathcal{Z}(z')$ in $\mathcal{N}_{n-1}^{[h]}$ and $(z', z') = (z, z)$.*
(2) *For $w \in \mathbb{V}$ such that $(x, w) = 0$, let $w' := (1 - e_x) \circ w$. Then, we have $\Phi(\mathcal{Z}(x) \cap \mathcal{Y}(w)) = \mathcal{Y}(w')$ in $\mathcal{N}_{n-1}^{[h]}$ and $(w', w') = (w, w)$.*
(3) *For $z \in \mathbb{V}$ such that $(y, z) = 0$, let $z' := (1 - e_y^{\vee}) \circ z$. Then, we have $\Psi(\mathcal{Y}(y) \cap \mathcal{Z}(z)) = \mathcal{Z}(z')$ in $\mathcal{N}_{n-1}^{[h-1]}$ and $(z', z') = (z, z)$.*
(4) *For $w \in \mathbb{V}$ such that $(y, w) = 0$, let $w' := (1 - e_y^{\vee}) \circ w$. Then, we have $\Psi(\mathcal{Y}(y) \cap \mathcal{Y}(w)) = \mathcal{Y}(w')$ in $\mathcal{N}_{n-1}^{[h-1]}$ and $(w', w') = (w, w)$.*

2.3. Horizontal and vertical part of special cycles. We closely follow [LZ22a, §2.9] in this subsection. We call a formal scheme Z over $\mathrm{Spf} O_{\check{F}}$ vertical (resp. horizontal) if π is locally nilpotent on Z (resp. flat over $\mathrm{Spf} O_{\check{F}}$). In particular, the formal scheme-theoretic union of two vertical (resp. horizontal) formal subschemes of a formal scheme is again vertical (resp. horizontal).

Now we define the horizontal part and vertical part of Z respectively. The horizontal part $Z_{\mathcal{H}}$ of Z is defined to be the closed formal subscheme with ideal sheaf $\mathcal{O}_Z[\pi^{\infty}] \subset \mathcal{O}_Z$. Then $Z_{\mathcal{H}}$ is the maximal horizontal closed formal subscheme of Z . For noetherian Z , we can find $N \gg 0$ such that $\pi^N \mathcal{O}_Z[\pi^{\infty}] = 0$. Then the vertical part $Z_{\mathcal{V}} \subset Z$ is defined to be the closed formal subscheme with ideal sheaf $\pi^N \mathcal{O}_Z$.

Note that $\mathcal{O}_Z[\pi^\infty] \cap \pi^N \mathcal{O}_Z = 0$ implies the following decomposition:

$$Z = Z_{\mathcal{H}} \cup Z_{\mathcal{Y}}.$$

The same proof of [LZ22a, Lemma 2.9.2] gives the following.

Lemma 2.8. [LZ22a, Lemma 2.9.2] *Let $L \subset \mathbb{V}$ be a O_F -lattice of rank $r \geq n - 1$ such that L_F is non-degenerate. Then $\mathcal{Z}(L)$ is noetherian.*

The following lemma follows from the same proof of [LZ22a, Lemma 5.1.1].

Lemma 2.9. [LZ22a, Lemma 5.1.1] *Let L^b be an O_F -lattice of rank $n - 1$ in \mathbb{V}_n . Then $\mathcal{Z}(L^b)_{\mathcal{Y}}$ is supported on $\mathcal{N}_n^{\text{red}}$, i.e., $\mathcal{O}_{\mathcal{Z}(L^b)_{\mathcal{Y}}}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}_n^{\text{red}} \subset \mathcal{N}_n$.*

2.4. Linear invariance. Following [How19], we show the linear invariance of intersection numbers. In this subsection, we use \mathcal{N} to denote $\mathcal{N}_n^{[h]}$ and (X, ι_X, λ_X) to denote the universal object over \mathcal{N} . Let $D(X)$ denote the covariant Grothendieck-Messing crystal of X restricted to the Zariski site. Then we have a short exact sequence of locally free $\mathcal{O}_{\mathcal{N}}$ -modules:

$$0 \rightarrow \text{Fil}(X) \rightarrow D(X) \rightarrow \text{Lie}(X) \rightarrow 0.$$

which is O_F -linear via the action given by ι_X . Let $O_F = O_{F_0} + O_{F_0}\alpha$ and

$$\begin{aligned} \epsilon &:= \bar{\alpha} \otimes 1 - 1 \otimes \bar{\alpha} \in O_F \otimes_{O_{F_0}} \mathcal{O}_{\mathcal{N}}, \\ \bar{\epsilon} &:= \alpha \otimes 1 - 1 \otimes \bar{\alpha} \in O_F \otimes_{O_{F_0}} \mathcal{O}_{\mathcal{N}}. \end{aligned}$$

Definition 2.10. Let L_X be the image of ϵ on $\text{Lie}(X)$. In other words, $L_X := \epsilon \text{Lie}(X)$.

According to the Kottwitz signature condition, we know $L_X \subset \text{Lie}(X)$ is locally a $\mathcal{O}_{\mathcal{N}}$ -module direct summand of rank 1.

For a closed formal subscheme Z of \mathcal{N} with ideal sheaf \mathcal{I}_Z , we denote by \tilde{Z} the closed formal subscheme defined by the sheaf \mathcal{I}_Z^2 . Let $x \in \mathbb{V}$ be a non-zero special homomorphism. Let X_0 be the universal object of $\mathcal{N}_1^{[0]}$. By the very definition of $\mathcal{Z}(x)$, we have

$$X_0|_{\mathcal{Z}(x)} \xrightarrow{x} X|_{\mathcal{Z}(x)},$$

which induces an O_F -linear morphism of vector bundles

$$D(X_0)|_{\mathcal{Z}(x)} \xrightarrow{x} D(X)|_{\mathcal{Z}(x)}.$$

By the Grothendieck–Messing theory, we may canonically extend the above morphism to a morphism

$$D(X_0)|_{\tilde{\mathcal{Z}}(x)} \xrightarrow{\tilde{x}} D(X)|_{\tilde{\mathcal{Z}}(x)},$$

which no longer preserves the Hodge filtrations and hence induces a nontrivial morphism

$$(2.2) \quad \text{Fil}(X_0)|_{\tilde{\mathcal{Z}}(x)} \xrightarrow{\tilde{x}} \text{Lie}(X)|_{\tilde{\mathcal{Z}}(x)}.$$

Proposition 2.11. *The morphism (2.2) induces a morphism*

$$\text{Fil}(X_0)|_{\tilde{\mathcal{Z}}(x)} \xrightarrow{\tilde{x}} L_X|_{\tilde{\mathcal{Z}}(x)}.$$

Moreover, $Z(x)$ is the vanishing locus of \tilde{x} .

Proof. According to the signature condition of X_0 , we have $\text{Fil}(X_0) = \epsilon D(X_0)$ since both are locally $\mathcal{O}_{\mathcal{N}}$ direct summands of $D(X_0)$ of rank 1 where $\iota(a)$ acts as \bar{a} . Since \tilde{x} is O_F -linear, we have

$$\tilde{x}(\text{Fil}(X_0)) = \tilde{x}(\epsilon D(X_0)) \in \epsilon \text{Lie}(X) = L_X.$$

Now the second claim follows from the Grothendieck–Messing theory. \square

Now given a nonzero element $x \in \mathbb{V}$, we define a chain complex of locally free $\mathcal{O}_{\mathcal{N}}$ -modules

$$C(x) := (\cdots \rightarrow 0 \rightarrow \mathcal{I}_{\mathcal{Z}(x)} \rightarrow \mathcal{O}_{\mathcal{N}} \rightarrow 0)$$

supported in degrees 1 and 0 with the map $\mathcal{I}_{\mathcal{Z}(x)} \rightarrow \mathcal{O}_{\mathcal{N}}$ being the natural inclusion. We extend the definition to $x = 0$ by setting

$$C(0) := (\cdots \rightarrow 0 \rightarrow \omega \xrightarrow{0} \mathcal{O}_{\mathcal{N}} \rightarrow 0)$$

supported in degrees 1 and 0, where ω is the line bundle such that $\omega^{-1} = \underline{\text{Hom}}(\text{Fil}(X_0), L_X)$.

The following is our main result of this subsection which follows from Proposition 2.12 by the same argument as in [How19].

Proposition 2.12. *Let $0 \leq m \leq n$ be an integer. Suppose that $x_1, \dots, x_m \in \mathbb{V}$ and $y_1, \dots, y_m \in \mathbb{V}$ generate the same O_E -submodule. Then we have an isomorphism*

$$H_i(C(x_1) \otimes_{\mathcal{O}_{\mathcal{N}}} \cdots \otimes_{\mathcal{O}_{\mathcal{N}}} C(x_m)) \simeq H_i(C(y_1) \otimes_{\mathcal{O}_{\mathcal{N}}} \cdots \otimes_{\mathcal{O}_{\mathcal{N}}} C(y_m))$$

of $\mathcal{O}_{\mathcal{N}}$ -modules for every i .

3. LOCAL DENSITY AND THE MODIFIED KUDLA–RAPOPORT CONJECTURE

In this section, we discuss the conjecture 1.3 proposed in the introduction in detail.

3.1. Local density. Let L, M be two integral hermitian O_F -lattices with rank n, m respectively. Let $\text{Herm}_{L, M}$ be the scheme of integral representations of M by L , an O_{F_0} -scheme such that for any O_{F_0} -algebra R ,

$$\text{Herm}_{L, M}(R) = \text{Herm}(L \otimes_{O_{F_0}} R, M \otimes_{O_{F_0}} R)$$

where Herm denotes the set of hermitian module homomorphisms. The local density of integral representations of M by L is defined to be

$$\text{Den}(M, L) := \lim_{d \rightarrow +\infty} \frac{\#\text{Herm}_{L, M}(O_{F_0}/(\pi^d))}{q^{d \cdot \dim(\text{Herm}_{L, M})_{F_0}}}.$$

If the generic fiber $(\text{Herm}_{L, M})_{F_0}$ is non-empty, then we have $n \leq m$ and

$$\dim(\text{Herm}_{M, L})_{F_0} = \dim U_m - \dim U_{m-n} = n \cdot (2m - n).$$

Now we consider the local density polynomial. Let I_k be an unimodular hermitian O_F -lattice of rank k . It is well-known that there exists a *local density polynomial* $\text{Den}(M, L, X) \in \mathbb{Q}[X]$ such that for any integer $k \geq 0$,

$$(3.1) \quad \text{Den}(M, L, (-q)^{-k}) = \text{Den}(I_k \oplus M, L).$$

When M has also rank n and $\chi(M) = -\chi(L)$, we have $\text{Den}(M, L) = 0$ and in this case we write

$$\text{Den}'(M, L) := -\frac{d}{dX} \Big|_{X=1} \text{Den}(M, L, X).$$

Define the (normalized) *derived local density*

$$(3.2) \quad \text{Den}'_{n,h}(L) := \frac{\text{Den}'(I_{n,h}, L)}{\text{Den}(I_{n,h}, I_{n,h})} \in \mathbb{Q}.$$

Here $I_{n,h}$ is a hermitian lattice with moment matrix $\text{Diag}((1)^{n-h}, (\pi)^h)$. When the n and h are clear in the context, we also simply denote it as $\text{Den}'(L)$.

Recall that an integral O_F -lattice $\Lambda \subset \mathbb{V}$ is a *vertex lattice of type t* if Λ^\vee/Λ is a κ -vector space of dimension t .

Lemma 3.1. *Assume $t < h$, we have $\text{Int}_{n,h}(\Lambda_t) = 0$.*

Proof. First, we write $\Lambda_t = I_{n-t} \oplus J_t$ where J_t is a lattice with moment matrix $(p)^t$. According to Proposition 2.6, $\mathcal{Z}(\Lambda_t)$ in $\mathcal{N}_n^{[h]}$ is isomorphic to $\mathcal{Z}(I_{n-t} \oplus J_t)$ in $\mathcal{N}_h^{[h]}$. However, for $(X, \iota, \lambda, \rho) \in \mathcal{N}_h^{[h]}$, we have $\text{Ker} \lambda = X[\pi]$ by definition of $\mathcal{N}_h^{[h]}$. Hence, $\lambda = \pi \lambda'$ for some principal λ' . Assume $(X, \iota, \lambda, \rho) \in \mathcal{Z}(x)$ for some $x \in \mathbb{V}$, then it is clear from the definition of $\mathcal{Z}(x)$ such that $\text{val}(x) \geq 1$. Hence, if $\text{val}(x) = 0$, then $\mathcal{Z}(x) = \emptyset$ in $\mathcal{N}_h^{[h]}$. In particular, $\mathcal{Z}(I_{n-t} \oplus J_t)$ in $\mathcal{N}_h^{[h]}$ is empty. \square

The naive analogue of the Kudla–Rapoport conjecture for $\mathcal{N}_n^{[h]}$ states that

$$\text{Int}_{n,h}(L) \stackrel{?}{=} \text{Den}'_{n,h}(L).$$

However, this can not be true since $\text{Int}_{n,h}(\Lambda_t) = 0$ by Lemma 3.1, but $\text{Den}'_{n,h}(\Lambda_t) \neq 0$.

Now a similar consideration as in [HSY23, HLSY23] suggests the following. In order to have $\text{Int}(\Lambda_t) = \partial \text{Den}(\Lambda_t)$ for vertex lattice Λ_t of type $t < h$, we define $\partial \text{Den}(L)$ by modifying $\text{Den}'(L)$ with a linear combination of the (normalized) *local densities*

$$(3.3) \quad \text{Den}_{n,t}(L) := \frac{\text{Den}(\Lambda_t, L)}{\text{Den}(\Lambda_t, \Lambda_t)} \in \mathbb{Z}.$$

Note that since F/F_0 is unramified, the hermitian space over F is determined by the parity of valuation of the hermitian form. As a result, if $\Lambda_t \subset \mathbb{V}$, then t and h have different parity.

Definition 3.2. Let $L \subset \mathbb{V}$ be an O_F -lattice. Define the *modified derived local density*

$$(3.4) \quad \partial \text{Den}_{n,h}(L) := \text{Den}'_{n,h}(L) + \sum_{i=0}^{\lfloor \frac{h-1}{2} \rfloor} c_{n,h-1-2i} \cdot \text{Den}_{n,h-1-2i}(L).$$

The coefficients $c_{n,i} \in \mathbb{Q}$ here are chosen to satisfy

$$(3.5) \quad \partial \text{Den}_{n,h}(\Lambda_i) = 0, \quad \text{for } 0 \leq i \leq h-1 \text{ and } i \equiv h-1 \pmod{2},$$

which turns out to be a linear system in $(c_{n,i})$ with a unique solution since $\text{Den}(\Lambda_j, \Lambda_i) = 0$ if $j > i$.

Conjecture 3.3. *Let $L \subset \mathbb{V}$ be an O_F -lattice. Then we have*

$$\text{Int}_{n,h}(L) = \partial \text{Den}_{n,h}(L).$$

Although the definition of $\partial\text{Den}(L)$ is very explicit, the computation of $\partial\text{Den}(L)$ is a challenging task, especially when $h > 0$, $I_{n,h}$ and the unimodular lattice I_k lie in two different Jordan block. One way to compute $\partial\text{Den}(L)$ is to decompose $\partial\text{Den}(L)$ into primitive pieces as we introduce now.

Similarly to the local density polynomial, we define the primitive local density polynomial $\text{Pden}(M, L, X)$ to be the polynomial in $\mathbb{Q}[X]$ such that

$$(3.6) \quad \text{Pden}(M, L, (-q)^{-k}) := \lim_{d \rightarrow +\infty} \frac{\#\text{Pherm}_{L,M}(O_{F_0}/(\pi^d))}{q^{d \cdot \dim(\text{Herm}_{L,M})_{F_0}}},$$

where

$$\text{Pherm}_{L,M \oplus H^k}(O_{F_0}/(\pi^d)) := \{\phi \in \text{Herm}_{L,M \oplus H^k}(O_{F_0}/(\pi^d)) \mid \phi \text{ is primitive}\}.$$

Recall that $\phi \in \text{Herm}_{L,M \oplus I_k}(O_{F_0}/(\pi^d))$ is primitive if $\dim_{\mathbb{F}_q}((\phi(L) + \pi(M \oplus I_k))/\pi(M \oplus I_k)) = n$. In particular, we have $\text{Den}(M, M) = \text{Pden}(M, M)$ for any hermitian O_F -lattice M . We can also similarly define the normalized primitive local densities:

$$\text{Pden}'_{n,h}(L) = \frac{\text{Pden}'(I_{n,h}, L)}{\text{Den}(I_{n,h}, I_{n,h})}, \quad \text{Pden}_{n,t}(L) := \frac{\text{Pden}(\Lambda_t, L)}{\text{Den}(\Lambda_t, \Lambda_t)},$$

and

$$\partial\text{Pden}_{n,h}(L) := \text{Pden}'_{n,h}(L) + \sum_{i=0}^{\lfloor \frac{h-1}{2} \rfloor} c_{n,h-1-2i} \cdot \text{Pden}_{n,h-1-2i}(L).$$

The following lemma decomposes local density polynomials into a summation of primitive local density polynomials. The following is essentially due to [CY20]. See also [LZ22a, Theorem 3.5.1].

Lemma 3.4. *Let M and L be lattices of rank m and n . Then we have*

$$\text{Den}(M, L, X) = \sum_{L \subset L' \subset L_F} (q^{n-m} X)^{\ell(L'/L)} \text{Pden}(M, L', X),$$

where $\ell(L'/L) = \text{length}_{O_F} L'/L$. Here $\text{Pden}(M, L', X) = 0$ for L' with fundamental invariant strictly less than the smallest fundamental invariant of M . In particular, the summation is finite.

Conversely, we can recover primitive local density polynomials as a linear combination of local density polynomials.

Theorem 3.5. [HSY23, Theorem 5.2] *Let M and L be lattices of rank m and n . We have*

$$\text{Pden}(M, L, X) = \sum_{i=0}^n (-1)^i q^{i(i-1)/2 + i(n-m)} X^i \sum_{\substack{L \subset L' \subset \pi^{-1}L \\ \ell(L'/L)=i}} \text{Den}(M, L', X).$$

Corollary 3.6. *Let L be a lattice of rank n . Then*

$$\partial\text{Pden}_{n,h}(L) = \sum_{i=0}^n (-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset \pi^{-1}L \\ \ell(L'/L)=i}} \partial\text{Den}_{n,h}(L').$$

Lemma 3.7. *For two lattices L and M of the same rank n , we have*

$$(3.7) \quad \text{Pden}(M, L) = \begin{cases} \text{Den}(M, L) & \text{if } M \cong L, \\ 0 & \text{if } M \not\cong L. \end{cases}$$

Moreover,

$$\text{Den}(M, L) = n(M, L) \cdot \text{Den}(M, M),$$

where for two lattices $M, L \subset \mathbb{V}$ of rank n , $n(M, L) = |\{L' \subset L_F \mid L \subset L', L' \cong M\}|$.

Corollary 3.8. *Assume $L \not\cong \Lambda_t$ for any vertex lattice Λ_t with $t < h$. Then*

$$\partial \text{Pden}_{n,h}(L) = \text{Pden}'_{n,h}(L).$$

Corollary 3.9. *Let $c_{n,t}$ be the coefficients in (3.5) with even t and $0 < t \leq t_{\max}$. Then*

$$c_{n,t} = -\text{Pden}'_{n,h}(\Lambda_t).$$

Proof. On the one hand, combining Corollary 3.6 with (3.5), we obtain

$$\partial \text{Pden}_{n,h}(\Lambda_t) = 0.$$

On the other hand, by Lemma 3.7 and (3.4),

$$\partial \text{Pden}_{n,h}(\Lambda_t) = \text{Pden}'_{n,h}(\Lambda_t) + c_{n,t}.$$

□

We will give another formulation of the conjecture 3.3 (conjecture 7.7) in §7 which is based on the duality between $\mathcal{N}_n^{[h]}$ and $\mathcal{N}_n^{[n-h]}$, and is in fact more general (it takes care of the intersections between \mathcal{Z} -cycles and \mathcal{Y} -cycles). The main terms of these two formulations are the same by direct calculations. However, interestingly, it is not clear that these two formulations have the same correction terms from the definition.

4. OUR STRATEGY: $\mathcal{N}_4^{[2]}$

Our general strategy is closest to the unramified unitary case [LZ22a] since $\mathcal{N}_n^{[0]} \subset \mathcal{N}_{n+h}^{[h]}$, and has several new ingredients which are quite complicated. Therefore, in this section, we consider the case $\mathcal{N}_4^{[2]}$ and explain our strategy. Indeed, this is the first new case we proved and almost all ideas are essentially from this case. Therefore, we believe that this section will be helpful for readers. Since we want to explain our strategy in detail, we will freely use notations from the following sections.

Let \mathbb{V} be the space of special homomorphisms (dimension 4), and let $L^b \subset \mathbb{V}$ be an O_F -lattice of rank 3. For any lattice L^b such that $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$, we define the primitive part $\mathcal{Z}(L^b)^\circ$ of the special cycle $\mathcal{Z}(L^b)$ inductively by setting

$$\mathcal{Z}(L^b)^\circ := \mathcal{Z}(L^b) - \sum_{\substack{L^b \subset L''^b \\ L''^b \subset (L''^b)^\vee \subset L_F^b}} \mathcal{Z}(L''^b)^\circ.$$

Then, for $x \in \mathbb{V} \setminus L_F^b$, the Kudla-Rapoport conjecture (Conjecture 7.6) is equivalent to

$$\text{Int}_{L^{b\circ}}(x) := \chi(\mathcal{N}_4^{[2]}, {}^{\mathbb{L}}\mathcal{Z}(L^b)^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) = \sum_{L^b \subset L' \subset L^\vee, L' \cap L_F^b = L^b} D_{4,2}(L') 1_{L'}(x) =: \partial \text{Den}_{L^{b\circ}}^{4,2}(x),$$

where $D_{4,2}(L')$ is the Cho-Yamauchi constant for $\mathcal{N}_4^{[2]}$ (Definition 8.3).

The first step is decomposing $\text{Int}_{L^{b\circ}}(x)$ and $\partial \text{Den}_{L^{b\circ}}^{4,2}(x)$ into horizontal and vertical parts:

$$\begin{aligned} \text{Int}_{L^{b\circ}}(x) &= \text{Int}_{L^{b\circ}, \mathcal{H}}(x) + \text{Int}_{L^{b\circ}, \mathcal{V}}(x), \\ \partial \text{Den}_{L^{b\circ}}^{4,2}(x) &= \partial \text{Den}_{L^{b\circ}, \mathcal{H}}^{4,2}(x) + \partial \text{Den}_{L^{b\circ}, \mathcal{V}}^{4,2}(x). \end{aligned}$$

In the case of the good reduction $\mathcal{N}_n^{[0]}$, this decomposition is relatively simple: if L^b has the fundamental invariants $(0, 0, \dots, 0, \alpha)$, $\alpha \geq 1$, then $\text{Int}_{L^{b\circ}}(x)$ and $\partial \text{Den}_{L^{b\circ}}^{4,2}(x)$ are horizontal. Otherwise, $\text{Int}_{L^{b\circ}}(x)$ and $\partial \text{Den}_{L^{b\circ}}^{4,2}(x)$ are vertical.

However, when $h \geq 1$, i.e., in the case of bad reduction, horizontal parts and vertical parts cannot be separated just by the fundamental invariants of L^b . Indeed, even in $\mathcal{N}_2^{[1]}$, $\mathcal{Z}(L_1)^\circ$ ($L_1 \simeq (\pi)$) is a sum of horizontal parts and vertical parts. To understand this phenomenon, we did some explicit computation on the analytic side by using [Cho22a]: for $L_{\beta\gamma\delta} \in \mathbb{V}$ of rank 3 with fundamental invariants (δ, γ, β) , and $x \in \mathbb{V} \setminus (L_{\beta\gamma\delta})_F$ with $\text{val}((x, x)) = \alpha \geq \beta \geq \gamma \geq \delta$, we have

$$\begin{aligned} \partial \text{Den}_{L_{100}^\circ}^{4,2}(x) &= \alpha/2, & \partial \text{Den}_{L_{300}^\circ}^{4,2}(x) &= (q^2 + q)\alpha/2 + 1 - q^2, \\ \partial \text{Den}_{L_{311}^\circ}^{4,2}(x) &= (q^7 + q^6)\alpha/2 - (q^7 - q^5 - 1), & \partial \text{Den}_{L_{210}^\circ}^{4,2}(x) &= q^3 - q + 1, \\ \partial \text{Den}_{L_{221}^\circ}^{4,2}(x) &= -(q^2 - 1)(q^2 - q + 1)(q^3 + q^2 + q + 1), \\ \partial \text{Den}_{L_{331}^\circ}^{4,2}(x) &= -(q^2 - 1)(q^6 + q^5 + q^4 + 2q^3 + q^2 + 1), \\ \partial \text{Den}_{L_{322}^\circ}^{4,2}(x) &= -(q^2 - 1)(q^2 - q + 1)(q^4 + 2q^3 + q^2 + q + 1), \\ & \vdots \end{aligned}$$

The most interesting thing in this computation is the fact that $\partial \text{Den}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x)$ does not depend on α if $(\beta, \gamma, \delta) \neq (\beta, 0, 0), (\beta, 1, 1)$. A similar phenomenon happens in the case of $\mathcal{N}_n^{[0]}$ for vertical components since these are locally constant. Therefore, it is reasonable to guess that the horizontal parts of $\mathcal{Z}(L^b)$ are contained in $\mathcal{Z}(L_{\beta\gamma\delta})^\circ$'s for $(\beta, \gamma, \delta) = (\beta, 0, 0), (\beta, 1, 1)$ (which turns out to be true by Theorem 5.3). Since $\mathcal{Z}(L_{\beta 00})^\circ, \mathcal{Z}(L_{\beta 11})^\circ$ have some vertical parts too, it should be handled carefully. Anyway, horizontal parts can be understood in this way, so for now, let us focus on the cases where $(\beta, \gamma, \delta) \neq (\beta, 0, 0), (\beta, 1, 1)$.

Now, we guessed that $\mathcal{Z}(L_{\beta\gamma\delta})^\circ$ is purely vertical if $(\beta, \gamma, \delta) \neq (\beta, 0, 0), (\beta, 1, 1)$. The next step is to understand the Fourier transforms

$$\begin{aligned} \widehat{\text{Int}}_{L_{\beta\gamma\delta}^\circ, \mathcal{V}}(x) &\stackrel{\text{guess}}{=} \widehat{\text{Int}}_{L_{\beta\gamma\delta}^\circ}(x), \\ \widehat{\partial \text{Den}}_{L_{\beta\gamma\delta}^\circ, \mathcal{V}}^{4,2}(x) &\stackrel{\text{guess}}{=} \widehat{\partial \text{Den}}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x), \end{aligned}$$

for $x \perp L_{\beta\gamma\delta}$, $\text{val}((x, x)) < 0$. In the case of good reduction $\mathcal{N}_n^{[0]}$, both $\widehat{\text{Int}}_{L^{b\circ}, \mathcal{V}}(x)$ and $\widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{V}}^{n,0}(x)$ vanish when $\text{val}((x, x)) < 0$, and hence

$$(4.1) \quad \widehat{\text{Int}}_{L^{b\circ}, \mathcal{V}}(x) - \widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{V}}^{n,0}(x) = 0, \quad \text{for } \text{val}((x, x)) < 0.$$

Since (4.1) is the most crucial property to prove the Kudla-Rapoport conjecture inductively, we have to show that (4.1) holds in our cases.

In $\mathcal{N}_4^{[2]}$, we still have that $\widehat{\partial\text{Den}}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x) = 0$ if $\text{val}((x, x)) \leq -2$, but $\widehat{\partial\text{Den}}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x)$ is not zero for $\text{val}((x, x)) = -1$ (see Theorem 10.18, Theorem 10.20, Theorem 10.19, Theorem 10.21, Theorem 11.2). Indeed, we can compute that for $\text{val}((x, x)) = -1$,

$$\begin{aligned}\widehat{\partial\text{Den}}_{L_{444}^\circ}^{4,2}(x) &= \frac{1}{q^2}(q^2 - 1)(q^3 + 1), & \widehat{\partial\text{Den}}_{L_{431}^\circ}^{4,2}(x) &= -\frac{1}{q^2}(q + 1)(q^3 - q + 1), \\ \widehat{\partial\text{Den}}_{L_{440}^\circ}^{4,2}(x) &= \frac{1}{q^2}(q^2 - 1), & \widehat{\partial\text{Den}}_{L_{310}^\circ}^{4,2}(x) &= -\frac{1}{q^2}.\end{aligned}$$

What is the meaning of these numbers? This is the main obstruction when we try to prove the conjecture. Since we cannot compute the geometric side directly, it is not possible to prove the conjecture without knowing the meaning of these numbers.

Fortunately, we had a table of the Cho-Yamauchi constants $D_{3,1}(L)$:

$$\begin{aligned}D_{3,1}(L_{444}) &= -(q^2 - 1)(q^3 + 1), & D_{3,1}(L_{431}) &= (q + 1)(q^3 - q + 1), \\ D_{3,1}(L_{440}) &= -(q^2 - 1), & D_{3,1}(L_{310}) &= 1.\end{aligned}$$

Now, it is easy to see that

$$(4.2) \quad \widehat{\partial\text{Den}}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x) = -\frac{1}{q^2}D_{3,1}(L_{\beta\gamma\delta}).$$

This is the most important observation in our work since this gives the following crucial ideas.

First, note that $\widehat{\partial\text{Den}}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x)$ is a certain linear sum of the Cho-Yamauchi constants $D_{4,2}(L)$ for $\mathcal{N}_4^{[2]}$ and the right-hand side of (4.2) is the Cho-Yamauchi constants $D_{3,1}(L)$ for $\mathcal{N}_3^{[1]}$. This suggests that there should be certain inductive relations among $D_{n,h}(L)$ and $D_{n-1,h-1}(L)$, $\forall 0 \leq h \leq n$ (see Theorem 9.4). Since the Cho-Yamauchi constants for $\mathcal{N}_n^{[h]}$ are very complicated (for example, $D_{6,2}(L) = (q-1)(q+1)^3(q^2-q+1)(q^{13}-q^{12}+q^{11}+q^{10}-2q^9+3q^8-3q^7+q^6-2q^4+2q^3-2q^2+q-1)$ for a lattice L with fundamental invariants $(1, 1, a_4, a_3, a_2, a_1)$, $a_i \geq 2$), we may not be able to find these inductive relations without the above observation (4.2).

These inductive relations are the most important ingredients to understand the analytic side and by using them, we have a quite complete understanding on the analytic side of the Kudla-Rapoport conjecture for $\mathcal{N}_n^{[h]}$, $\forall 0 \leq h \leq n$.

Second, note that the right-hand side of (4.2) is the Cho-Yamauchi constant for $\mathcal{N}_3^{[1]}$, not $\mathcal{N}_3^{[2]}$. Also, note that we need a Y -cycle to get a reduction from $\mathcal{N}_4^{[2]}$ to $\mathcal{N}_3^{[1]}$ (see Proposition 2.6). This means that Y -cycles appear when we take the Fourier transform of $\text{Int}_{L_{\beta\gamma\delta}^\circ}(x) = \chi(\mathcal{N}_4^{[2]}, \mathbb{L}\mathcal{Z}(L_{\beta\gamma\delta}^\circ) \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$. Indeed, the Kudla-Rapoport conjecture for $\mathcal{N}_3^{[1]}$ holds, therefore,

$$D_{3,1}(L_{\beta\gamma\delta}) = \chi(\mathcal{N}_4^{[2]}, \mathbb{L}\mathcal{Z}(L_{\beta\gamma\delta}^\circ) \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}), \quad \text{val}((x, x)) = -1.$$

Therefore, (4.2) suggests that

$$\begin{aligned}\chi(\mathcal{N}_4^{[2]}, \widehat{\text{Int}}_{L_{\beta\gamma\delta}^\circ}(x) \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) &= \widehat{\text{Int}}_{L_{\beta\gamma\delta}^\circ}(x) \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)} \\ &\stackrel{\text{conjecture}}{=} \widehat{\partial\text{Den}}_{L_{\beta\gamma\delta}^\circ}^{4,2}(x) = -\frac{1}{q^2}D_{3,1}(L_{\beta\gamma\delta}) = -\frac{1}{q^2}\chi(\mathcal{N}_4^{[2]}, \mathbb{L}\mathcal{Z}(L_{\beta\gamma\delta}^\circ) \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}).\end{aligned}$$

If we can prove this relation, then we can show that (4.1) holds for all $\text{val}(x, x) < 0$, and we can use an inductive argument to prove the Kudla-Rapoport conjecture for $\mathcal{N}_4^{[2]}$.

By [LZ22a, Lemma 6.3.1] and [Zha22, Theorem 8.1], we know that

$$\chi(\widehat{\mathcal{N}_4^{[2]}, O_C \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}}) = -\frac{1}{q^2} \chi(\mathcal{N}_4^{[2]}, O_C \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}),$$

if C is a Deligne-Lusztig curve or \mathbb{P}^1 . Therefore, if $\mathcal{Z}(L_{\beta\gamma\delta})^\circ$ is a linear sum of Deligne-Lusztig curves or \mathbb{P}^1 (in the Grothendieck group of coherent sheaves), then we have

$$\chi(\widehat{\mathcal{N}_4^{[2]}, {}^{\mathbb{L}}\mathcal{Z}(L_{\beta\gamma\delta})^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}}) = -\frac{1}{q^2} \chi(\mathcal{N}_4^{[2]}, {}^{\mathbb{L}}\mathcal{Z}(L_{\beta\gamma\delta})^\circ \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}).$$

This is how we make the Conjecture 6.3 (this can be regarded as a variant of Tate conjectures for certain Deligne-Lusztig varieties), and we prove that if Conjecture 6.3 holds, then the Kudla-Rapoport conjecture holds (see Theorem 11.4). Then, we prove that Conjecture 6.3 holds for $\mathcal{N}_4^{[2]}$ and some other cases (see Theorem 6.5). This is how we prove the Kudla-Rapoport conjecture for $\mathcal{N}_4^{[2]}$.

5. HORIZONTAL PARTS OF KUDLA-RAPOPORT CYCLES

In this section, we describe the horizontal parts of special cycles following the approach of [LZ22a, §4]. Due to the existence of non-trivial level structures, there are some new phenomena.

Let K denote a finite extension of \check{F} . Consider $z \in \mathcal{N}_n^{[h]}(O_K)$ which corresponds to an O_F -hermitian module G of signature $(1, n-1)$ over O_K . Let $T_p(-)$ denote the integral p -adic Tate modules and

$$L := \text{Hom}_{O_F}(T_p \bar{\mathcal{E}}, T_p G).$$

We can associate L with a hermitian form $\{x, y\}$ given by

$$(T_p \bar{\mathcal{E}} \xrightarrow{x} T_p G \xrightarrow{\lambda_G} T_p G^\vee \xrightarrow{y^\vee} T_p \bar{\mathcal{E}}^\vee \xrightarrow{\lambda_{\bar{\mathcal{E}}}^\vee} T_p \bar{\mathcal{E}}) \in \text{End}_{O_F}(T_p \bar{\mathcal{E}}) \cong O_F.$$

One can check that L is represented by the hermitian matrix $\text{Diag}((1)^{n-h}, (\pi)^h)$.

Following [LZ22a, §4], we consider two injective O_F -linear isometric homomorphisms

$$i_K : \text{Hom}_{O_F}(\bar{\mathcal{E}}, G)_F \rightarrow L_F,$$

and

$$i_{\bar{k}} : \text{Hom}_{O_F}(\bar{\mathcal{E}}, G)_F \rightarrow \mathbb{V}.$$

By [LZ22a, Lemma 4.4.1], we have

$$(5.1) \quad \text{Hom}_{O_F}(\bar{\mathcal{E}}, G) = i_K^{-1}(L).$$

By the definition of special cycles, for any O_F -lattice $M \subset \mathbb{V}$, we have $z \in \mathcal{Z}(M)(O_K)$ if and only if $M \subset i_{\bar{k}}(\text{Hom}_{O_F}(\bar{\mathcal{E}}, G))$. By (5.1), $z \in \mathcal{Z}(M)(O_K)$ if and only if

$$(5.2) \quad M \subset i_{\bar{k}}(i_K^{-1}(L)).$$

Now assume that $z \in \mathcal{Z}(L^b)(O_K)$ corresponds to an O_F -hermitian module G of signature $(1, n-1)$ over O_K . By (5.2), we have

$$L^b \subset i_{\bar{k}}(i_K^{-1}(L)).$$

Define $M^b := L_F^b \cap i_{\bar{k}}^{-1}(i_K^{-1}(L))$. By (5.2), we still have $z \in \mathcal{Z}(M^b)(O_K)$. Moreover, we have

$$M^b \xrightarrow{\sim} L \cap i_K(i_{\bar{k}}^{-1}(L_F^b)).$$

Set $\mathbb{W} = i_K(i_{\bar{k}}^{-1}(L_F^b))$, which has the same dimension as L_F^b .

Definition 5.1. Define $H(\mathbb{V})$ to be the collection of O_F -lattices $M^b \subset \mathbb{V}$ of rank $n - 1$ such that $M^b \approx \text{Diag}((1)^{n-h}, (\pi)^{h-2}, (\pi^a))$ with $a \in \mathbb{Z}_{>0}$ or $M^b \approx \text{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a))$ with $a \in \mathbb{Z}_{\geq 0}$.

Definition 5.2. For any lattice L^b such that $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$, we define the primitive part $\mathcal{Z}(L^b)^\circ$ of the special cycle $\mathcal{Z}(L^b)$ inductively by setting

$$\mathcal{Z}(L^b)^\circ := \mathcal{Z}(L^b) - \sum_{\substack{L'^b \subset L^b \\ L'^b \subset (L'^b)^\vee \subset L_F^b}} \mathcal{Z}(L'^b)^\circ.$$

Moreover, we define $\mathcal{Y}(L^b)^\circ$ similarly.

Theorem 5.3. Let $L^b \subseteq \mathbb{V}_n$ be a hermitian O_F -lattice of rank $n - 1$. Then

$$\mathcal{Z}(L^b)_{\mathcal{H}} = \bigcup_{\substack{L^b \subseteq M^b, \\ M^b \in H(\mathbb{V})}} \mathcal{Z}(M^b)_{\mathcal{H}}^\circ.$$

Moreover, we have the following.

- (1) If $M^b \approx \text{Diag}((1)^{n-h}, (\pi)^{h-2}, (\pi^a))$ with $a \in \mathbb{Z}_{>0}$, then $\mathcal{Z}(M^b)_{\mathcal{H}} \cong \mathcal{Z}(M^b) \simeq \mathcal{Z}(x) \subset \mathcal{N}_2^{[0]}$ with $\text{val}(x) = a - 1$, which is a quasi-canonical lifting of degree $a - 1$.
- (2) If $M^b \approx \text{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a))$ with $a \in \mathbb{Z}_{\geq 0}$, then

$$\mathcal{Z}(M^b)_{\mathcal{H}}^\circ = \sum_{\substack{M^b \subset M_1 \oplus N_2 \subset \pi^{-1}M^b \\ N_2 \approx (\pi^{-1})^h}} \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ.$$

Here, each $\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ$ is a quasi-canonical lifting of degree a . The summation index has cardinality:

$$\begin{cases} q^{2n-2} & \text{if } a \geq 1, \\ q^{2n-2} \frac{1 - (-q)^{-n}}{1 - (-q)^{-1}} & \text{if } a = 0. \end{cases}$$

The rest of this section is devoted to the proof of Theorem 5.3. We give the proof at the end of this section after some preparations.

First, by the similar method as in the appendix of [HSY23], we can prove the following two lemmas.

Lemma 5.4. Assume $L = L_1 \oplus L_2$ where $L_1 \approx (1)^{n-h}$ and $L_2 \approx (\pi)^h$. Let x be a primitive vector in L .

- (1) If $\text{Pr}_{L_1}(x)$ is primitive in L_1 , then there exists $L'_1 \approx (1)^{n-h}$ such that $x \in L'_1$.
- (2) If $\text{Pr}_{L_1}(x)$ is not primitive in L_1 , then there exists $L'_1 \approx (1)^{n-h}$ and $L'_2 \approx (\pi)^h$ such that $L = L'_1 \oplus L'_2$ and $x \in L'_2$.

Lemma 5.5. *Assume $L \approx (1)^{n-h}$ or $(\pi)^h$. Then for any primitive vectors $x, x' \in L$ with $q(x) = q(x')$, there exists $g \in U(L)$ such that $g(x) = x'$.*

With the help of the above two lemmas, we can prove the following.

Lemma 5.6. *Assume L is a hermitian lattice represented by $\text{Diag}((1)^{n-h}, (\pi)^h)$ and $\mathbb{W} \subset L_F$ is a subspace of dimension $n-1$. Then $M^b := \mathbb{W} \cap L$ is represented either by $\text{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a))$ with $a \in \mathbb{Z}_{\geq 0}$ or $\text{Diag}((1)^{n-h}, (\pi)^{h-2}, (\pi^a))$ with $a \in \mathbb{Z}_{>0}$. In the first case, we can write $M^b = M_1 \oplus M_2$ such that $L = L_1 \oplus M_2$.*

Proof. First, we assume $M_1^b := M^b \cap L_{1,F}$ is not unimodular. Then the rank of L_1 is at least 2 and we choose a basis $\{e_1, \dots, e_{n-h}\}$ of L_1 such that $e_1, e_2 \in L_1$ and $(e_1, e_1) = (e_2, e_2) = 0$, $(e_1, e_2) = 1$ and $(e_i, e_j) = 0$ for $i, j > 2, i \neq j$. Since M_1^b is primitive in L_1 but non-isometric to L_1 , we know the rank of M_1^b is smaller than $n-h$. Hence the rank of M_1^b has to be $n-h-1$ and $M_2^b := M^b \cap L_{2,F} = L_2$. Now we choose a orthogonal basis $\{x_1, \dots, x_{n-h-1}\}$ of M_1^b , where we assume $q(x_1) = (x_1, x_1)$ has largest valuation among $\{q(x_1), \dots, q(x_{n-h-1})\}$. In particular, $\text{val}(q(x_1)) > 0$. By Lemmas 5.4 and 5.5, we may assume $x_1 = e_1 + \frac{q(x_1)}{2}e_2$. Since M_1^b is primitive, we can write $x_i = a_i(e_1 - \frac{q(x_1)}{2}e_2) + \sum_{j=3}^{n-h} a_{ij}e_j$ and for each j with $3 \leq j \leq n-h$, there exists an i such that a_{ij} is a unit. Hence by possibly choosing a different basis $\{x_2, \dots, x_{n-h-1}\}$, we may assume $x_i = a_i(e_1 - \frac{q(x_1)}{2}e_2) + e_{i+1}$. Now it is clear that $\text{Span}\{x_2, \dots, x_{n-h-1}\}$ is unimodular and $M^b \approx \text{Diag}((1)^{n-h-2}, (\pi)^h, (\pi^a))$.

Now we assume $M_1^b := M^b \cap L_{1,F}$ is unimodular. If $\text{rank} M_1^b = n-h-1$, then an argument as above shows that $M^b \approx \text{Diag}((1)^{n-h-1}, (\pi)^h)$. If $\text{rank} M_1^b = n-h$, then $\text{rank} M_2^b = h-1$, and a similar argument as before shows that $M_2^b \approx (\pi)^{h-2} \oplus (\pi^a)$ with $a \in \mathbb{Z}_{>0}$. Then $M^b \approx (1)^{n-h} \oplus (\pi)^{h-2} \oplus (\pi^a)$ and $(1)^{n-h-1} \oplus (\pi)^h$ respectively. \square

In particular, Lemma 5.6 implies that if $z \in \mathcal{Z}(L^b)(O_K)$, then $z \in \mathcal{Z}(M^b)(O_K)$ where $M^b \in H(\mathbb{V})$.

Now we assume $z \in \mathcal{Z}(L^b)(O_K)$ corresponds to a M^b represented by $(1)^{n-h-2} \oplus (\pi^a) \oplus (\pi)^h$. By Proposition 2.7, we may reduce to the case M^b represented by $(\pi^a) \oplus (\pi)^h$.

Proposition 5.7. *Assume $z \in \mathcal{Z}(L^b)(O_K)$ corresponds to a $M^b = M_1 \oplus M_2$, where M_1 is represented by (π^a) and M_2 is represented by $(\pi)^h$. If $L = L_1 \oplus M_2$, then $z \in \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ$.*

Proof. First, notice that $z \in \mathcal{Z}(M^b)^\circ$ essentially by the definition of M^b . We may choose a basis $\{x_1, x_2\}$ of L_1 and a basis $\{x_3, \dots, x_{h+2}\}$ of M_2 such that the moment matrix of $\{x_1, x_2\}$ and $\{x_3, \dots, x_{h+2}\}$ is $(1)^2$ and $(\pi)^h$. Let

$$L_{z^\vee} := \text{Hom}_{O_F}(T_p \bar{\mathcal{E}}, T_p G^\vee).$$

Composing L with λ_G , we obtain an embedding $L \hookrightarrow L_{z^\vee}$ such that $L_{z^\vee} = \lambda_G(L_1) \oplus \pi^{-1}\lambda_G(M_2)$. Moreover, the moment matrix of $\lambda_G(L_1) \oplus \pi^{-1}\lambda_G(M_2)$ is $(\pi)^2 \oplus (1)^h$. We can also directly check that $L_{z^\vee} \cap \lambda_G(i_K(i_k^{-1}(L_F^b))) = M_1 \oplus \pi^{-1}\lambda_G(M_2)$. Hence, $z \in \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ$ by the definition of special cycles. \square

Lemma 5.8. *Assume $M^b = M_1 \oplus M_2 = M_1' \oplus M_2$. Then*

$$\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ = \mathcal{Z}(M_1')^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ.$$

Proof. First of all, we have $\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ \subset \mathcal{Z}(M_1)^\circ \cdot \mathcal{Z}(M_2)^\circ = \mathcal{Z}(M'_1)^\circ \cdot \mathcal{Z}(M_2)^\circ \subset \mathcal{Z}(M'_1)^\circ$. Hence, $\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ \subset \mathcal{Z}(M'_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ$. By switching the role of M_1 and M'_1 , the lemma is proved. \square

Lemma 5.9. *Let $M^b = M_1 \oplus M_2$, where $M_1 = \text{span}\{x_0\}$, and M_2 has a basis $\{x_1, \dots, x_h\}$ with moment matrix $(\pi)^h$. Assume $M^b \subset N^b \subset \pi^{-1}M^b$ and $N^b \approx \text{Diag}(\pi^a, (\pi^{-1})^h)$. Then $N^b = M_1 \oplus \text{span}\{\pi^{-1}(x_1 + \alpha_1 x_0), \dots, \pi^{-1}(x_h + \alpha_h x_0)\}$, where α_i is a representative of $O_F/(\pi)$.*

Proof. Since $M^b \approx \text{Diag}(\pi^a, (\pi)^h)$ and $N^b \approx \text{Diag}(\pi^a, (\pi^{-1})^h)$, N^b must contain a sub-lattice of the form $\text{span}\{\pi^{-1}(x_1 + \alpha_1 x_0), \dots, \pi^{-1}(x_h + \alpha_h x_0)\}$. Moreover, we have $M^b \stackrel{h}{\subset} N^b \subset \pi^{-1}M^b$.

Notice that $M^b \stackrel{h}{\subset} M_1 \oplus \text{span}\{\pi^{-1}(x_1 + \alpha_1 x_0), \dots, \pi^{-1}(x_h + \alpha_h x_0)\} \subset N^b$. Hence $N^b = M_1 \oplus \text{span}\{\pi^{-1}(x_1 + \alpha_1 x_0), \dots, \pi^{-1}(x_h + \alpha_h x_0)\}$. \square

Lemma 5.10. *Assume $M^b = M_1 \oplus M_2$ with basis $\{x_0, x_1, \dots, x_h\}$ as in Lemma 5.9. Let N^b and $(N^b)'$ be two different lattices such that $M^b \subset N^b \subset \pi^{-1}M^b$, $M^b \subset (N^b)' \subset \pi^{-1}M^b$, and $N^b \approx (N^b)' \approx \text{Diag}(\pi^a, (\pi^{-1})^h)$. Write $N^b = M_1 \oplus N_2$ and $(N^b)' = M_1 \oplus N'_2$. Then*

$$\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ(O_K) \neq \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N'_2)^\circ(O_K).$$

Proof. Let $N^b = M_1 \oplus \text{span}\{\pi^{-1}(x_1 + \alpha_1 x_0), \dots, \pi^{-1}(x_h + \alpha_h x_0)\}$ and $(N^b)' = M_1 \oplus \text{span}\{\pi^{-1}(x_1 + \alpha'_1 x_0), \dots, \pi^{-1}(x_h + \alpha'_h x_0)\}$ where α_i and α'_i are representatives of $O_F/(\pi)$. If $\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ(O_K) = \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N'_2)^\circ(O_K)$, then we have nontrivial $z \in \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ(O_K) \cap \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N'_2)^\circ(O_K)$. This implies that $z \in \mathcal{Y}(\pi^{-1}M_1)$. In particular, $z \in \mathcal{Y}(\pi^{-1}M_1) \cdot \mathcal{Y}(\text{span}\{\pi^{-1}x_1, \dots, \pi^{-1}x_h\})(O_K)$. However, notice that

$$\mathcal{Y}(\pi^{-1}M_1) \cdot \mathcal{Y}(\text{span}\{\pi^{-1}x_1, \dots, \pi^{-1}x_h\}) \cong \mathcal{Z}(\pi^{-1}M_1) \cdot \mathcal{Y}(\text{span}\{\pi^{-1}x_1, \dots, \pi^{-1}x_h\})$$

by cancellation law and the fact that $\mathcal{Z}(\pi^{-1}M_1) = \mathcal{Y}(\pi^{-1}M_1)$ in \mathcal{N}_2 . This contradicts the fact $z \notin \mathcal{Z}(\pi^{-1}M_1)^\circ$. \square

Proof of Theorem 5.3. Assume $z \in \mathcal{Z}(M^b)^\circ(O_K)$. By Lemma 5.6, we know that $z \in \mathcal{Z}(M^b)(O_K)$ where $M^b \in H(\mathbb{V})$.

If $M^b = M_1 \oplus M_2$, where $M_1 \approx (1)^{n-h}$ and $M_2 \approx \text{Diag}((\pi)^{h-2}, (\pi^a))$ with $a > 0$, then according to Proposition 2.7, $\mathcal{Z}(M^b) \cong \mathcal{Z}(M_2) \subset \mathcal{N}_h^{[h]}$. Then applying the duality between $\mathcal{N}_h^{[h]}$ and $\mathcal{N}_h^{[0]}$, we have $\mathcal{Z}(M_2) \subset \mathcal{N}_h^{[h]}$ is isomorphic to $\mathcal{Z}(M'_2) \subset \mathcal{N}_h^{[0]}$, where $M'_2 \cong \text{Diag}((1)^{h-2}, (\pi^{a-1}))$. Then by Proposition 2.7 and [KR11, Proposition 8.1], $\mathcal{Z}(M^b)^\circ$ is isomorphic to a quasi-canonical lifting of degree $a - 1$.

Now we assume $M^b \approx \text{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a))$ with $a \in \mathbb{Z}_{\geq 0}$. Then according to Proposition 5.7, and Lemma 5.9, we have $z \in \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ(O_K)$ for some N_2 , where $N_2 = \text{span}\{\pi^{-1}(x_1 + \alpha_1 x_0), \dots, \pi^{-1}(x_h + \alpha_h x_0)\}$ and each α_i is some representative of $O_F/(\pi)$. If $M_1 \cong \text{Diag}(1^{n-h-2}, \pi^a)$ and $M_2 \cong (\pi)^h$, then by Proposition 2.7 and [KR11, Proposition 8.1] again, $\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1}M_2)^\circ$ is isomorphic to a quasi-canonical lifting of degree a .

According to the above discussion, we have

$$(5.3) \quad \mathcal{Z}(L^b)_{\mathcal{H}} = \bigcup_{\substack{L^b \subset M^b, \\ M^b \in H(\mathbb{V})}} m(M^b) \mathcal{Z}(M^b)_{\mathcal{H}}^\circ,$$

and $\mathcal{Z}(M^b)_{\mathcal{H}}^{\circ}$ has the desired description.

Now we show $m(M^b) = 1$ for any M^b in the above identity following the proof of [LZ22a, Theorem 4.2.1] closely. It suffices to show $O_K[\epsilon]$ -points of both sides of (5.3) are the same where $\epsilon^2 = 0$. First, each O_K -point of the right hand side of (5.3) has a unique lift to an $O_K[\epsilon]$ -point. Therefore we only need to show each $z \in \mathcal{Z}(M^b)_{\mathcal{H}}^{\circ}(O_K)$ has a unique lift in $\mathcal{Z}(M^b)_{\mathcal{H}}^{\circ}(O_K[\epsilon])$.

Let G be the corresponding O_F -hermitian module of signature $(1, n-1)$ over O_K and $\mathbb{D}(G)$ be the (covariant) O_{F_0} -relative Dieudonné crystal of G . First, we have an action $O_F \otimes_{O_{F_0}} O_K \simeq O_K \oplus O_K$ on $\mathbb{D}(G)(O_K)$ induced by the action of O_F via $\iota : O_F \rightarrow \text{End}(G)$, and hence a $\mathbb{Z}/2\mathbb{Z}$ -grading on $\mathbb{D}(G)(O_K)$. Then we let $\mathcal{A} = \text{gr}_0 \mathbb{D}(G)(O_K)$. It is a free O_K -module of rank n equipped with an O_K -hyperplane: $\mathcal{H} := \text{Fil}^1 \mathbb{D}(G)(O_K) \cap \mathcal{A}$ by the Kottwitz signature condition. Note that \mathcal{H} contains the image of L^b under the identification of [KR11, Lemma 3.9].

Note that the kernel of $O_K[\epsilon] \rightarrow O_K$ has a natural nilpotent divided power structure. Then according to Grothendieck-Messing theory, a lift $\tilde{z} \in \mathcal{Z}(L^b)(O_K[\epsilon])$ of z corresponds to an $O_K[\epsilon]$ direct summand of $\mathbb{D}(G)(O_K[\epsilon])$ $\widetilde{\text{Fil}}$ that lifts $\text{Fil}^1 \mathbb{D}(G)$ and contains the image of L^b in $\widetilde{\mathcal{A}}$. Here, $\widetilde{\text{Fil}}$ is isotropic under the natural pairing $\langle \cdot, \cdot \rangle_{\mathbb{D}(G)(O_K[\epsilon])}$ on $\mathbb{D}(G)(O_K[\epsilon])$ induced by the polarization. Since $L^b \subset \text{Hom}_{O_F}(\overline{\mathcal{E}}, G)$ has rank $n-1$, by Breuil's theorem [LZ22a, §4.3], we know that the image of L^b in $\text{gr}_0 \mathbb{D}(G)(S)$ has rank $n-1$ over S (the Breuil's ring) and thus its image in the base change \mathcal{A} has rank $n-1$ over O_K . In particular, $\text{gr}_0 \widetilde{\text{Fil}}$ is the unique $O_K[\epsilon]$ -hyperplane $\widetilde{\mathcal{H}}$ of $\text{gr}_0 \mathbb{D}(G)(O_K[\epsilon])$ that contains the $O_K[\epsilon]$ -module spanned by the image of L^b in $\widetilde{\mathcal{A}}$.

To determine $\text{gr}_1 \widetilde{\text{Fil}}$, we note that $\widetilde{\text{Fil}}$ is a direct summand of $\mathbb{D}(G)(O_K[\epsilon])$ with rank n containing $\widetilde{\mathcal{H}}$. Since $\widetilde{\text{Fil}}$ is isotropic under $\langle \cdot, \cdot \rangle_{\mathbb{D}(G)(O_K[\epsilon])}$, we have $\text{gr}_1 \widetilde{\text{Fil}} \subset (\widetilde{\mathcal{H}})^{\perp} \cap \text{gr}_1 \mathbb{D}(G)(O_K[\epsilon])$. Here $(\widetilde{\mathcal{H}})^{\perp}$ is the perpendicular subspace in $\mathbb{D}(G)(O_K[\epsilon])$ with respect to $\langle \cdot, \cdot \rangle_{\mathbb{D}(G)(O_K[\epsilon])}$. Moreover, since $\det \langle \cdot, \cdot \rangle_{\mathbb{D}(G)(O_K[\epsilon])} \neq 0$ in $O_K[\epsilon]$, we have $(\widetilde{\mathcal{H}})^{\perp}$ has rank $n+1$. Note that $\text{gr}_0 \mathbb{D}(G)(O_K[\epsilon])$ is also isotropic under $\langle \cdot, \cdot \rangle_{\mathbb{D}(G)(O_K[\epsilon])}$. In particular, $\text{gr}_0 \mathbb{D}(G)(O_K[\epsilon]) \subset (\widetilde{\mathcal{H}})^{\perp}$ which has rank n . Hence $(\widetilde{\mathcal{H}})^{\perp} \cap \text{gr}_1 \mathbb{D}(G)(O_K[\epsilon])$ is of rank one. Since $\widetilde{\text{Fil}}$ is a direct summand of $\mathbb{D}(G)(O_K[\epsilon])$, we know $\text{gr}_1 \widetilde{\text{Fil}} = (\widetilde{\mathcal{H}})^{\perp} \cap \text{gr}_1 \mathbb{D}(G)(O_K[\epsilon])$. Hence $\widetilde{\text{Fil}} = \text{gr}_0 \widetilde{\text{Fil}} \oplus \text{gr}_1 \widetilde{\text{Fil}}$ is uniquely determined, and the lift $\tilde{z} \in \mathcal{Z}(L^b)(O_K[\epsilon])$ of z is unique. Hence $\widetilde{\text{Fil}} = \text{gr}_0 \widetilde{\text{Fil}} \oplus \text{gr}_1 \widetilde{\text{Fil}}$ is uniquely determined, and the lift $\tilde{z} \in \mathcal{Z}(L^b)(O_K[\epsilon])$ of z is unique.

We defer the proof of the cardinality of the summation index to Lemma 10.22, where it is proved via analytic method. \square

6. LOCAL MODULARITY AND TATE CONJECTURES

First, as we have discussed in the introduction, we propose the following local modularity conjecture motivated by the analytic computation and the special case for $\mathcal{N}_n^{[0]}$ (see [LZ22a, Corollary 5.3.3]).

Conjecture 6.1. *For the Rapoport-Zink space $\mathcal{N}_n^{[h]}$ and an O_F -lattice $L^b \subset \mathbb{V}$ of rank $n-1$, we have*

$$\widehat{\text{Int}}_{L^b, \mathcal{Y}}(\mathcal{Z}(x)) = -\frac{1}{q^h} \text{Int}_{L^b, \mathcal{Y}}(\mathcal{Y}(x)).$$

Remark 6.2. More precisely, as we wrote in Section 4, Conjecture 6.1 is primarily motivated by (4.2) and, more generally, by Theorem 11.2 below along with [LZ22a, Lemma 6.3.1] and [Zha22, Theorem 8.1].

Conjecture 6.3. (cf. [LZ22a, Corollary 5.3.3]) *For the Rapoport-Zink space $\mathcal{N}_n^{[h]}$ and an O_F -lattice $L^\flat \subset \mathbb{V}$ of rank $n - 1$, there are finitely many Deligne-Lusztig curves $C_i \subset \mathcal{N}_3^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, projective lines $D_i \subset \mathcal{N}_2^{[1]} \hookrightarrow \mathcal{N}_n^{[h]}$ and $\text{mult}_{C_i}, \text{mult}_{D_i} \in \mathbb{Q}$ such that for any $x \in \mathbb{V} \setminus L_F^\flat$,*

$$\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathcal{V}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) = \sum_i \text{mult}_{C_i} \chi(\mathcal{N}_n^{[h]}, O_{C_i} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) + \sum_i \text{mult}_{D_i} \chi(\mathcal{N}_n^{[h]}, O_{D_i} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}).$$

Remark 6.4. Indeed, as we wrote in Section 4, Conjecture 6.1 follows from Conjecture 6.3. Conjecture 6.3 is implied by a stronger version of Tate conjectures on 1-cycles for Deligne-Lusztig varieties Y_Λ in Proposition 6.6 (cf. [LZ22a, Theorem 5.3.2]).

Theorem 6.5. *Conjecture 6.3 holds for $\mathcal{N}_n^{[0]}, \mathcal{N}_n^{[1]}, \mathcal{N}_n^{[n-1]}, \mathcal{N}_n^{[n]}$, and $\mathcal{N}_4^{[2]}$.*

To prove this theorem, we follow [LZ22a, Section 5.3]. First, we need to recall several notations and theorems from [Cho18]. In [Cho18, Theorem 1.1], we proved that the reduced subscheme of $\mathcal{N}_n^{[h]}$ has a Bruhat-Tits stratification and their components are certain Deligne-Lusztig varieties Y_Λ where Λ is a vertex lattice of type $t(\Lambda)$. More precisely, we have the following proposition.

Proposition 6.6. [Cho18, Theorem 1.1] *Let $\mathcal{N}_{n,\text{red}}^{[h]}$ be the underlying reduced subscheme of $\mathcal{N}_n^{[h]}$. Then, we have*

$$\mathcal{N}_{n,\text{red}}^{[h]} = \bigcup_{t(\Lambda) \leq h-1} Y_\Lambda \cup \bigcup_{t(\Lambda) \geq h+1} Y_\Lambda,$$

where Y_Λ denotes certain Deligne-Lusztig varieties associated with vertex lattices Λ . Also, the dimension of Y_Λ is $\frac{1}{2}(t(\Lambda) + h - 1)$ (resp. $\frac{1}{2}(t(\Lambda) + n - h - 1)$) if $t(\Lambda) \geq h + 1$ (resp. $t(\Lambda) \leq h - 1$).

By [LZ22a, Corollary 5.3.3], we know that the Conjecture 6.3 holds for $\mathcal{N}_n^{[0]}$ and $\mathcal{N}_n^{[n]}$. Also, by [Cho18, Theorem 1.1], the irreducible components of the reduced subscheme of $\mathcal{N}_4^{[2]}$ are \mathbb{P}^2 and their Chow groups are well-known. Therefore, let us focus on $\mathcal{N}_n^{[1]}, \mathcal{N}_n^{[n-1]}$. Since $\mathcal{N}_n^{[1]}$ is isomorphic to $\mathcal{N}_n^{[n-1]}$, we only need to consider $\mathcal{N}_n^{[1]}$. In this case, by Proposition 6.6 (see [Cho18, Theorem 1.1] for more detail), we know that the reduced subscheme of $\mathcal{N}_n^{[1]}$ has a Bruhat-Tits stratification and their components are Deligne-Lusztig varieties Y_Λ where $t(\Lambda) \geq 2$ or projective spaces \mathbb{P}_Λ^n , where $t(\Lambda) = 0$. Let us describe Y_Λ more precisely.

Let k_F be the residue field of F and let V_{2d+h+1} be the unique (up to isomorphism) k_F/k -hermitian space of dimension $2d + h + 1$. Let $\Lambda/\pi\Lambda^\vee = V_{2d+h+1}$ where $t(\Lambda) = 2d + h + 1$, and let J_{2d+h+1} be the special unitary group associated to $(V_{2d+h+1}, (\cdot, \cdot))$. Let (W_{2d+h+1}, S_{2d+h+1}) be the Weyl system of J_{2d+h+1} and let B_{2d+h+1} be the standard Borel subgroup. Let $\mathcal{F} : J_{2d+h+1} \rightarrow J_{2d+h+1}$ be the Frobenius morphism over $k = \mathbb{F}_q$. For $I \subset S_{2d+h+1}$, we define W_I as the subgroup of W_{2d+h+1} generated by I and $P_I := B_{2d+h+1} W_I B_{2d+h+1}$. Note that W_{2d+h+1} can be identified with a symmetric group S_{2d+h+1} with $\{s_1, \dots, s_{2d+h}\}$ where s_i is the transposition of i and $i + 1$.

We write

$$\begin{aligned}
I_0 &:= \begin{cases} \{s_1, \dots, s_d, s_{d+2}, \dots, s_{2d+1}\}, & \text{if } h = 1 \\ \{s_1, \dots, s_d, s_{d+2}, \dots, s_{d+h}, s_{d+h+2}, \dots, s_{2d+h}\}, & \text{if } h > 1 \end{cases}, \\
I_i &:= \begin{cases} \{s_1, \dots, s_{d-i}, s_{d+i+2}, \dots, s_{2d+1}\}, 1 \leq i \leq d, & \text{if } h = 1 \\ \{s_1, \dots, s_{d-i}, s_{d+2}, \dots, s_{d+h}, s_{d+h+i+2}, \dots, s_{2d+h}\}, 1 \leq i \leq d & \text{if } h > 1 \end{cases}, \\
P_i &:= P_{I_i}, \\
w_\Lambda &:= s_{d+1}s_{d+2} \dots s_{d+h}.
\end{aligned}$$

Also, for $I \subset S_{2d+h+1}$ and $w \in W_{2d+h+1}$, we define the Deligne-Lusztig variety $X_I(w)$ by

$$X_I(w) := \{g \in J_{2d+h+1}/P_I | g^{-1}\mathcal{F}(g) \in P_I w P_{\mathcal{F}(I)}\}.$$

Finally, we define Y_Λ by $Y_\Lambda := \overline{X_{I_0}(w_\Lambda)} = X_{I_0}(id) \sqcup X_{I_0}(w_\Lambda)$ (see [Cho18, Definition 3.10]). Then, we have the following proposition.

Proposition 6.7. *Y_Λ is smooth.*

Proof. Note that Y_Λ is smoothly equivalent to the closure of $P_0 w_\Lambda \mathcal{F}(P_0) / \mathcal{F}(P_0) \subset J_{2d+h+1} / \mathcal{F}(P_0)$, a Schubert variety in the partial flag variety $J_{2d+h+1} / \mathcal{F}(P_0)$ ([GHN24, Section 7.1]). Also, this is smoothly equivalent to its inverse image in J_{2d+h+1} / B_{2d+h+1} , the Schubert variety for the maximal element in $P_0 w_\Lambda \mathcal{F}(P_0)$ ([GHN24, Section 7.2]). Now, by [LS90, Theorem 1] or [BL00, Theorem 8.1.1], it suffices to show that the maximal element in $P_0 w_\Lambda \mathcal{F}(P_0)$ avoids the patterns 3412, 4231. By the balls-in-boxes picture for w_Λ ([BKP⁺18, Section 3.2]), we can compute that the maximal element in $P_0 w_\Lambda \mathcal{F}(P_0)$ is

$$\begin{pmatrix} 1 & 2 & \dots & d & d+1 & \dots & d+h & d+h+1 & d+h+2 & \dots & 2d+h & 2d+h+1 \\ d+1 & d & \dots & 2 & d+h+1 & \dots & d+2 & 2d+h+1 & 2d+h & \dots & d+h+2 & 1 \end{pmatrix}.$$

It is now easy to see that this element avoids the patterns 3412, 4231, and hence Y_Λ is smooth. \square

Now, assume that $h = 1$. In this case, we have that $P_d = B_{2d+2}$. Also, note that the elements in J_{2d+2}/P_i parametrize flags

$$0 \subset T_{-i} \subset T_{-i+1} \dots \subset T_{-1} \subset \overline{A} \subset \overline{B} \subset T_1 \dots \subset T_i \subset V_{2d+2}.$$

Now, by [Cho18, Theorem 1.1], we know that the reduced subscheme of $\mathcal{N}_n^{[1]}$ is

$$(6.1) \quad \mathcal{N}_{n,red}^{[1]} = \cup_{\Lambda, t(\Lambda) \geq 2} Y_\Lambda \cup \cup_{\Lambda, t(\Lambda) = 0} \mathbb{P}_\Lambda^n.$$

Here, $Y_\Lambda = X_{P_0}(id) \sqcup X_{P_0}(s_{d+1})$ where $2d+2 = t(\Lambda)$ (see [Cho18, Definition 3.10]). By [Cho18, Lemma 2.21] (cf. [Vol10, Lemma 2.1]), we have the following statement (cf. [Vol10, Theorem 2.15]).

Proposition 6.8. (cf. [Vol10, Theorem 2.15]) *There is a decomposition of $X_{P_0}(id) \sqcup X_{P_0}(s_{d+1})$ into a disjoint union of locally closed subvarieties*

$$X_{P_0}(id) \sqcup X_{P_0}(s_{d+1}) = \sqcup_{i=0}^d \{X_{P_i}(s_{d+2} \dots s_{d+i+1}) \sqcup X_{P_i}(s_{d+1} s_{d+2} \dots s_{d+i+1})\}.$$

Proof. We can follow the proof of [Vol10, Theorem 2.15] with [Cho18, Lemma 2.21]. \square

Since $P_d = B_{2d+2}$, we have that $X_{P_d}(s_{d+2} \dots s_{2d+1})$ and $X_{P_d}(s_{d+1}s_{d+2} \dots s_{2d+1})$ are classical Deligne-Lusztig varieties. For $0 \leq i \leq d$, let us write

$$\begin{aligned} X_{i,1}^\circ &:= X_{P_i}(s_{d+1} \dots s_{d+i+1}), \\ X_{i-1,2}^\circ &:= X_{P_i}(s_{d+2} \dots s_{d+i+1}), \\ Y_{i,1}^\circ &:= X_{B_{2i+2}}(s_{i+1} \dots s_{2i+1}), \\ Y_{i-1,2}^\circ &:= X_{B_{2i+2}}(s_{i+2} \dots s_{2i+1}), \\ \widetilde{X}_i^\circ &:= X_{i,1}^\circ \sqcup X_{i,2}^\circ, \\ \widetilde{X}_i &:= \sqcup_{m=0}^i \widetilde{X}_m^\circ, \\ Y_d &:= \sqcup_{i=0}^d \{X_{P_i}(s_{d+2} \dots s_{d+i+1}) \sqcup X_{P_i}(s_{d+1}s_{d+2} \dots s_{d+i+1})\}. \end{aligned}$$

Then, by the above proposition and a Bruhat-Tits stratification in [Cho18, Theorem 1.1], we have that $X_{i,1}^\circ$ (resp. $X_{i,2}^\circ$) is a disjoint union of isomorphic copies of $Y_{i,1}^\circ$ (resp. $Y_{i,2}^\circ$). Also, the dimensions of $Y_{i,1}^\circ$ and $Y_{i,2}^\circ$ are $i+1$.

For any k_F -variety S , we write $H^j(S)(i)$ for $H^j(S_{\bar{k}}, \overline{\mathbb{Q}}_l(i))$ where $l \neq p$ is a prime and \bar{k} is an algebraically closed field containing k_F . Let $\mathcal{F} = \text{Fr}_{k_F}$ be the q^2 -Frobenius on $H^j(S)(i)$. Then, the following analogous statement of [LZ22a, Lemma 5.3.1] holds.

Lemma 6.9. (cf. [LZ22a, Lemma 5.3.1]) *For any $d, i \geq 0$ and $s \geq 1$, the action of \mathcal{F}^s on the following cohomology groups are semisimple, and the space of \mathcal{F}^s -invariants is zero when $j \geq 1$.*

- (1) $H^{2j}(Y_{d,1}^\circ)(j)$.
- (2) $H^{2j}(Y_{d-1,2}^\circ)(j)$.
- (3) $H^{2j}(\widetilde{X}_i^\circ)(j)$.
- (4) $H^{2j}(Y_d - \widetilde{X}_i)(j)$.

Proof. Here, we follow the proof of [LZ22a, Lemma 5.3.1] with some modification.

- (1) By [Lus76, (7.3) (${}^2A_{2d+1}, \mathcal{F}$)] (or we refer to the proof of [Ohm10, Lemma 2]), we have the following table on the eigenvalues of \mathcal{F} on $H_c^j(Y_{d,1}^\circ)$.

j	$d+1$	$d+2$	$d+3$	\dots	$2d-1$	$2d$	$2d+1$	$2d+2$
	1	q^2	q^4	\dots	q^{2d-4}	q^{2d-2}	q^{2d}	q^{2d+2}
	$-q^3$	$-q^5$	$-q^7$	\dots	$-q^{2d-1}$.

By the Poincaré duality, we have a perfect pairing

$$H_c^{2d+2-j}(Y_{d,1}^\circ) \times H^j(Y_{d,1}^\circ)(d+1) \rightarrow H_c^{2d+2}(Y_{d,1}^\circ)(d+1) \simeq \overline{\mathbb{Q}}_l.$$

Therefore, the eigenvalues of \mathcal{F} on $H^{2j}(Y_{d,1}^\circ)(j)$ are given by $q^{2(d+1-j)}$ times the inverse of the eigenvalues in $H_c^{2(d+1-j)}(Y_{d,1}^\circ)$. More precisely,

$$\begin{array}{ccc} H_c^{2(d+1-j)}(Y_{d,1}^\circ) & & q^{2(d+1-j)} \times \text{the inverse} \\ j = 0 & q^{2d+2} & 1 \\ j = 1 & q^{2d-2} & q^2 \\ j \geq 2 & q^{2d+2-4j}, -q^{2d-4j+5} & q^{2j}, -q^{2j-3}. \end{array}$$

Therefore, the eigenvalue of \mathcal{F}^s cannot be 1 when $j \geq 1$. The semisimplicity of the action of \mathcal{F}^s is from [Lus76, 6.1].

(2) By [Lus76, (7.3) (A_{d+1}, \mathcal{F})] (or we refer to the proof of [Ohm10, Lemma 2], here, note that \mathcal{F} is q^2 -Frobenius), we have the following table on the eigenvalues of \mathcal{F} on $H_c^j(Y_{d,2}^\circ)$.

j	$d+1$	$d+2$	$d+3$	\dots	$2d-1$	$2d$	$2d+1$	$2d+2$
	1	q^2	q^4	\dots	q^{2d-4}	q^{2d-2}	q^{2d}	q^{2d+2}

By the Poincaré duality, we have a perfect pairing

$$H_c^{2d+2-j}(Y_{d,2}^\circ) \times H^j(Y_{d,2}^\circ)(d+1) \rightarrow H_c^{2d+2}(Y_{d,2}^\circ)(d+1) \simeq \overline{\mathbb{Q}}_l.$$

Therefore, the eigenvalues of \mathcal{F} on $H^{2j}(Y_{d,2}^\circ)(j)$ are given by $q^{2(d+1-j)}$ times the inverse of the eigenvalues in $H_c^{2(d+1-j)}(Y_{d,2}^\circ)$. More precisely,

	$H_c^{2(d+1-j)}(Y_{d,2}^\circ)$	$q^{2(d+1-j)} \times \text{the inverse}$
$j=0$	q^{2d+2}	1
$j=1$	q^{2d-2}	q^2
$j \geq 2$	$q^{2d+2-4j}$	q^{2j} .

Therefore, the eigenvalue of \mathcal{F}^s cannot be 1 when $j \geq 1$. The semisimplicity of the action of \mathcal{F}^s is from [Lus76, 6.1].

(3) This follows from (1) and (2) since \widetilde{X}_i° is a disjoint union of $Y_{i,1}^\circ$ and $Y_{i,2}^\circ$.

(4) This follows from (3) since $Y_d - \widetilde{X}_i^\circ = \sqcup_{m=i+1}^d \widetilde{X}_m^\circ$.

□

Theorem 6.10. (cf. [LZ22a, Theorem 5.3.2]) *For any $0 \leq i \leq d+1$ and any $s \geq 1$, we have*

- (1) *The space of Tate classes $H^{2i}(Y_d)(i)^{\mathcal{F}^s=1}$ is spanned by the cycle classes of the irreducible components of \widetilde{X}_i .*
- (2) *Let $H^{2i}(Y_d)(i)_1 \subset H^{2i}(Y_d)(i)$ be the generalized eigenspace of \mathcal{F}^s for the eigenvalue 1. Then $H^{2i}(Y_d)(i)_1 = H^{2i}(Y_d)(i)^{\mathcal{F}^s=1}$.*

Proof. The proof is the same as [LZ22a, Theorem 5.3.2] with Lemma 6.9

□

Proof of Theorem 6.5. Here, we follow the proof of [LZ22a, Corollary 5.3.3]. For $\mathcal{N}_n^{[0]}$ and $\mathcal{N}_n^{[n]}$, this is from [LZ22a, Corollary 5.3.3]. For $\mathcal{N}_n^{[1]}$ and $\mathcal{N}_n^{[n-1]}$, note that $\mathcal{N}_n^{[1]}$ and $\mathcal{N}_n^{[n-1]}$ are isomorphic, so we only need to consider the case $\mathcal{N}_n^{[1]}$. By [Cho18, Theorem 1.1], we have (6.1):

$$\mathcal{N}_{n,red}^{[1]} = \cup_{\Lambda, t(\Lambda) \geq 2} Y_\Lambda \cup \cup_{\Lambda, t(\Lambda)=0} \mathbb{P}_\Lambda^n,$$

and any curve C in $\mathcal{N}_{n,red}^{[1]}$ lies in some $Y_\Lambda \simeq Y_d$ or the projective space \mathbb{P}_Λ^n for some vertex lattice Λ . If it lies on Y_d , then by Theorem 6.10, the cycle class of C can be written as a \mathbb{Q} -linear combination of the cycle classes of the irreducible components of \widetilde{X}_1 and these are projective lines. Similarly, if it lies on \mathbb{P}_Λ^n , then by the Chow group of \mathbb{P}_Λ^n , we have that the cycle class of C can be written as a \mathbb{Q} -linear combination of projective lines. This finishes the proof of the theorem for $\mathcal{N}_n^{[1]}$ and $\mathcal{N}_n^{[n-1]}$.

For $\mathcal{N}_4^{[2]}$, we know that the irreducible components of $\mathcal{N}_{4,red}^{[2]}$ are \mathbb{P}_Λ^2 for vertex lattices Λ , and hence by the description of Chow group of \mathbb{P}_Λ^2 , the cycle class of C can be written as a \mathbb{Q} -linear

combination of projective lines. Moreover, the projective lines here can be realized as images of embeddings of projective lines in $\mathcal{N}_2^{[1]}$.

Finally, the finiteness of curves is from Lemma 2.8. This finishes the proof of the theorem. \square

7. WEIGHTED REPRESENTATION DENSITIES AND CONJECTURES

In this section, we first recall the definition of weighted representation densities and formulas from [Cho22b, Section 3.1]. Then, we will recall the conjectural formula in [Cho22b, Conjecture 3.17, Conjecture 3.25].

We denote by $*$ the nontrivial Galois automorphism of F over F_0 . We fix the standard additive character $\psi : F_0 \rightarrow \mathbb{C}^\times$ that is trivial on O_{F_0} . Let V^+ (resp. V^-) be a split (resp. non-split) $2n$ -dimensional hermitian vector space over F and let $\mathcal{S}((V^\pm)^{2n})$ be the space of Schwartz functions on $(V^\pm)^{2n}$. Let $V_{r,r}$ be the split hermitian space of signature (r, r) and let $L_{r,r}$ be a self-dual lattice in $V_{r,r}$. Let $\phi_{r,r}$ be the characteristic function of $(L_{r,r})^{2n}$. Let $(V^\pm)^{[r]}$ be the space $V^\pm \otimes V_{r,r}$. For any function $\phi \in \mathcal{S}((V^\pm)^{2n})$, we define a function $\phi^{[r]}$ by $\phi \otimes \phi_{r,r} \in \mathcal{S}(((V^\pm)^{[r]})^{2n})$.

Let Γ_n be the Iwahori subgroup

$$\Gamma_n := \{\gamma = (\gamma_{ij}) \in GL_n(O_F) \mid \gamma_{ij} \in \pi O_F \text{ if } i > j\}.$$

We define the set $V_n(F)$ by

$$V_n(F) = \{Y \in M_{n,n}(F) \mid {}^t Y^* = Y\}.$$

We define the set $X_n(F)$ by

$$X_n(F) = \{X \in GL_n(F) \mid {}^t X^* = X\}.$$

For $g \in GL_n(F)$ and $X \in X_n(F)$, we define the group action of $GL_n(F)$ on $X_n(F)$ by $g \cdot X = gX^t g^*$. For $X, Y \in V_n(F)$, we denote by $\langle X, Y \rangle = \text{Tr}(XY)$. For $X \in M_{m,n}(F)$ and $A \in V_m(F)$, we denote by $A[X] = {}^t X^* A X$. For a hermitian matrix $A \in X_m(F)$, we define

$$A^{[r]} = \begin{pmatrix} A & \\ & I_{2r} \end{pmatrix}.$$

Now, let us recall the definition of usual representation densities.

Definition 7.1. For $A \in X_m(O_F)$ and $B \in X_n(O_F)$, we define $\text{Den}(A, B)$ by

$$\text{Den}(A, B) = \lim_{d \rightarrow \infty} (q^{-d})^{n(2m-n)} |\{x \in M_{m,n}(O_F/\pi^d O_F) \mid A[x] \equiv B \pmod{\pi^d}\}|.$$

Now, let us recall the definition of weighted representation densities in [Cho22b].

Definition 7.2. [Cho22b, Definition 3.1] Let $0 \leq h, t \leq n$. Let L_t be a lattice of rank $2n$ in V^+ if t is even (resp. in V^- if t is odd) with hermitian form

$$A_t := \begin{pmatrix} I_{2n-t} & \\ & \pi^{-1} I_t \end{pmatrix}.$$

Let $1_{h,t} \in \mathcal{S}((V^\pm)^{2n})$ be the characteristic function of $(L_t^\vee)^{2n-h} \times L_t^h$.

For $B \in X_{2n}(F)$, we define

$$W_{h,t}(B, (-q)^{-2r}) := \int_{V_{2n}(F)} \int_{M_{2n+2r, 2n}(F)} \psi(\langle Y, A_t^{[r]}[X] - B \rangle) 1_{h,t}^{[r]}(X) dX dY.$$

Here, dY (resp. dX) is the Haar measure on $V_{2n}(F)$ (resp. $M_{2n+2r, 2n}(F)$) such that

$$\int_{V_{2n}(O_F)} dY = 1 \quad (\text{resp.} \quad \int_{M_{2n+2r, 2n}(O_F)} dX = 1).$$

The functions $\text{Den}(A, B)$ and $W_{h,t}(B, r)$ have the following formulas.

Lemma 7.3. ([Hir00], [Cho22b, Lemma 3.5]) *For $A \in X_m(F)$ and $B \in X_{2n}(F)$, we have that*

$$\text{Den}(A^{[r]}, B) = \sum_{Y \in \Gamma_{2n} \backslash X_{2n}(F)} \frac{\mathcal{G}(Y, B) \mathcal{F}(Y, A^{[r]})}{\alpha(Y; \Gamma_{2n})},$$

and

$$W_{h,t}(B, (-q)^{-2r}) = \sum_{Y \in \Gamma_{2n} \backslash X_{2n}(F)} \frac{\mathcal{G}(Y, B) \mathcal{F}_h(Y, A_t^{[r]})}{\alpha(Y; \Gamma_{2n})}.$$

Here, we define $\mathcal{F}(Y, A^{[r]})$ by

$$\mathcal{F}(Y, A^{[r]}) := \int_{M_{m, 2n}(F)} \psi(\langle Y, A^{[r]}[X] \rangle) dX,$$

and we define $\mathcal{F}_h(Y, A_t^{[r]})$ by

$$\mathcal{F}_h(Y, A_t^{[r]}) := \int_{M_{2n+2r, 2n}(F)} \psi(\langle Y, A_t^{[r]}[X] \rangle) 1_{h,t}^{[r]}(X) dX.$$

We define $\mathcal{G}(Y, B)$ by

$$\mathcal{G}(Y, B) := \int_{\Gamma_{2n}} \psi(\langle Y, -B[\gamma] \rangle) d\gamma,$$

where $d\gamma$ is the Haar measure on $M_{2n, 2n}(O_F)$ such that $\int_{M_{2n, 2n}(O_F)} d\gamma = 1$.

Also, we define $\alpha(Y; \Gamma_{2n})$ by

$$\alpha(Y; \Gamma_{2n}) := \lim_{d \rightarrow \infty} q^{-4dn^2} N_d(Y; \Gamma_{2n}),$$

where $N_d(Y; \Gamma_{2n}) = |\{\gamma \in \Gamma_{2n}(\text{mod } \pi^d) \mid \gamma \cdot Y \equiv Y(\text{mod } \pi^d)\}|$.

Definition 7.4. For $r \geq 0$, $A \in X_{n+2r}(O_F)$, and $B \in X_n(O_F)$, we can regard $\mathcal{F}(Y, A^{[r]})$, $\text{Den}(A^{[r]}, B)$, $\mathcal{F}_h(Y, A_t^{[r]})$, and $W_{h,t}(B, (-q)^{-2r})$ as functions of $X = (-q)^{-2r}$. We define

$$\mathcal{F}'(Y, A) := -\frac{d}{dX} \mathcal{F}(Y, A^{[r]})|_{X=1},$$

and

$$\mathcal{F}'_h(Y, A_t) := -\frac{d}{dX} \mathcal{F}_h(Y, A_t^{[r]})|_{X=1}.$$

Also, we define

$$\text{Den}'(A, B) = -\frac{d}{dX} \text{Den}(A, B; X)|_{X=1},$$

and

$$W'_{h,t}(B) := -\frac{d}{dX} W_{h,t}(B, r)|_{X=1}.$$

Definition 7.5. [Cho23, Proposition 2.7] For $0 \leq i, h \leq 2n$, we define the constant β_i^h by

$$\beta_i^h = \alpha_{i+1,h}^{-1} \left(\frac{\prod_{1 \leq m \leq 2n, m \neq i+1} (1 - x_m)}{\prod_{1 \leq m \leq 2n+1, m \neq i+1} (x_m - x_{i+1})} \right),$$

where

$$\begin{aligned} \alpha_{i,h} &= (-q)^{(n+1-i)(2n-h)}, & 1 \leq i \leq n; \\ \alpha_{i,h} &= (-q)^{(2n+1-i)(2n+h)}, & n+1 \leq i \leq 2n; \\ \alpha_{2n+1,h} &= 1, \end{aligned}$$

and

$$\begin{aligned} x_i &= (-q)^{n+1-i}, & 1 \leq i \leq n; \\ x_i &= (-q)^{i-2n-1}, & n+1 \leq i \leq 2n; \\ x_{2n+1} &= 1. \end{aligned}$$

Now, we can state [Cho22b, Conjecture 3.17, Conjecture 3.25].

Conjecture 7.6. [Cho22b, Conjecture 3.17, Conjecture 3.25]

For a basis $\{x_1, \dots, x_{2n-m}, y_1, \dots, y_m\}$ of \mathbb{V} , and special cycles $\mathcal{Z}(x_1), \dots, \mathcal{Z}(x_{2n-m}), \mathcal{Y}(y_1), \dots, \mathcal{Y}(y_m)$ in $\mathcal{N}_{2n}^{[n]}$, we have

$$\chi(\mathcal{N}_{2n}^{[n]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} O_{\mathcal{Y}(y_m)}) = \frac{1}{W_{n,n}(A_n, 1)} \{W'_{m,n}(B) - \sum_{0 \leq i \leq n-1} \beta_i^m W_{m,i}(B, 1)\}.$$

Here, χ is the Euler-Poincaré characteristic and $\otimes^{\mathbb{L}}$ is the derived tensor product. Also, B is the matrix

$$B = \left(\begin{array}{cc} (x_i, x_j) & (x_i, y_l) \\ (y_k, x_j) & (y_k, y_l) \end{array} \right)_{\substack{1 \leq i, j \leq 2n-m, \\ 1 \leq k, l \leq m}}.$$

Assume that special homomorphisms $\{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+h}, y_1, \dots, y_{n-h}\}$ has the hermitian matrix:

$$(7.1) \quad B = \left(\begin{array}{cc} (x_i, x_j) & (x_i, y_l) \\ (y_k, x_j) & (y_k, y_l) \end{array} \right)_{\substack{1 \leq i, j \leq n+h, \\ 1 \leq k, l \leq n-h}} = \begin{pmatrix} T & & \\ & I_h & \\ & & \pi^{-1} I_{n-h} \end{pmatrix},$$

for some $n \times n$ matrix T . Then, by Proposition 2.6 and Proposition 2.7, the arithmetic intersection number of special cycles $\chi(\mathcal{N}_{2n}^{[n]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_{n+h})} \otimes^{\mathbb{L}} O_{\mathcal{Y}(y_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} O_{\mathcal{Y}(y_{n-h})})$ in $\mathcal{N}_{2n}^{[n]}$ can be identified with $\text{Int}_{n,h}(T) = \text{Int}_{n,h}(L) = \chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_n)})$ in $\mathcal{N}_n^{[h]}$, where $L = \text{Span}_{O_F}\{x_1, \dots, x_n\}$. We note that the valuation of the determinant of B and $h+1$ have the same parity. Now, Conjecture 7.6 is specialized to the following conjecture.

Conjecture 7.7. Consider a basis $\{x_1, \dots, x_{n+h}, y_1, \dots, y_{n-h}\}$ of \mathbb{V} with moment matrix B as in (7.1). Let $L = \text{Span}_{O_F}\{x_1, \dots, x_n\}$. Then

$$(7.2) \quad \text{Int}_{n,h}(L) = \text{Int}_{n,h}(T) = \frac{1}{W_{n,n}(A_n, 1)} \{W'_{n-h,n}(B) - \sum_{0 \leq i \leq n-1} \beta_i^{n-h} W_{n-h,i}(B, 1)\}.$$

Note that $\text{Int}_{n,h}(L)$ is exactly the intersection number considered in Conjecture 3.3. We show the analytic sides of Conjectures 3.3 and 7.7 also match in §8 (see Corollary 8.7).

8. CHO-YAMAUCHI CONSTANTS

In this section, we will modify the result of [Cho23] to get the Cho-Yamauchi constants in the case of $\mathcal{N}_n^{[h]}$. More precisely, we want to write the conjectural formula for the arithmetic intersection numbers of special cycles $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_n)})$ in $\mathcal{N}_n^{[h]}$ as a linear sum of representation densities. First, we start with the following proposition.

Proposition 8.1. *Assume that B is of the form in (7.1). Then, we have*

$$\begin{aligned} W'_{n-h,n}(B) &= q^{-4n^2+(n+h)(n-h)} \text{Den}(\pi A_n, I_{n-h}) \text{Den}'(I_{n+h,h}, \begin{pmatrix} T & \\ & I_h \end{pmatrix}) \\ &= q^{-4n^2+(n+h)(n-h)} \text{Den}(\pi A_n, I_{n-h}) \text{Den}(I_{n+h,h}, I_h) \text{Den}'(I_{n,h}, T). \end{aligned}$$

Proof. One can use a similar method as in [KR11, Corollary 9.12] to prove this. For example, see [Cho22b, Proposition A.3, Proposition A.4, (A.0.4), (A.0.5)]. \square

Note that in Proposition 8.1, the terms $\text{Den}(\pi A_n, I_{n-h})$ and $\text{Den}(I_{n+h,h}, I_h)$ are constants, and $\text{Den}'(I_{n,h}, T)$ is the derivative of a usual representation density. Now, by [Cho23], this can be written as a linear sum of usual representation densities. Let us follow the steps in [Cho23, Section 4.1]. For this, we need to introduce some notations.

Definition 8.2. (1) We write \mathcal{R}_n for the set

$$\mathcal{R}_n = \{Y_{\sigma,e} \mid (\sigma, e) \in \mathcal{S}_n \times \mathbb{Z}^n, \sigma^2 = 1, e_i = e_{\sigma(i)}, \forall i\},$$

where \mathcal{S}_n is the symmetric group of degree n , and

$$Y_{\sigma,e} = \sigma \begin{pmatrix} \pi^{e_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{e_n} \end{pmatrix}.$$

Then \mathcal{R}_n forms a complete set of representatives of $\Gamma_n \backslash X_n(F)$.

(2) We write \mathcal{R}_n^{0+} for the set

$$\mathcal{R}_n^{0+} = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

(3) For $\lambda \in \mathcal{R}_n^{0+}$, we define A_λ by

$$A_\lambda = \begin{pmatrix} \pi^{\lambda_1} & & \\ & \ddots & \\ & & \pi^{\lambda_n} \end{pmatrix}.$$

(4) For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{R}_n^{0+}$, we define $|\lambda|$ by

$$|\lambda| = \sum_{i=1}^n \lambda_i.$$

Definition 8.3 (Cho-Yamauchi constant). Assume that B is of the form in (7.1). For $\lambda \in \mathcal{R}_n^{0+}$, we define $D_{n,h}(\lambda)$ to be the constant satisfying

$$\frac{1}{W_{n,n}(A_n, 1)} \left\{ W'_{n-h,n}(B) - \sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1) \right\} = \sum_{\lambda \in \mathcal{R}_n^{0+}} D_{n,h}(\lambda) \frac{\text{Den}(A_\lambda, T)}{\text{Den}(A_\lambda, A_\lambda)}.$$

The existence and uniqueness of these constants are from [Cho23]. The constant $D_{n,h}(\lambda)$ is a version of the Cho-Yamauchi constant in [CY20].

Remark 8.4. In $\mathcal{N}^h(1, n-1)$, let L be a rank n O_F -lattice generated by special homomorphisms x_1, \dots, x_n in \mathbb{V} . Assume that T is the hermitian matrix of L . Then, the valuation of the determinant of T and $h+1$ have the same parity. Therefore, in Definition 8.3, the terms $\text{Den}(A_\lambda, T)$ such that

$$\text{val}(\det(A_\lambda)) = \sum_i \lambda_i \not\equiv h+1 \pmod{2}$$

are always equal to 0.

$$\text{Now, let us compute the correction terms } \frac{- \sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1)}{W_{n,n}(A_n, 1)}.$$

Proposition 8.5. Assume that B is of the form in (7.1). If $n-h \leq t \leq n-1$, we have

$$W_{n-h,t}(B, 1) = q^{-4n^2+(n+h)(3n-2t-h)} \text{Den}(\pi A_t, I_{n-h}) \text{Den}(I_{n+h,t-n+h}, I_h) \text{Den}(I_{n,t-n+h}, T).$$

If $t < n-h$, we have that $W_{n-h,t}(B, 1) = 0$.

Proof. One can use a similar method as in [KR11, Corollary 9.12] to prove this. For example, see [Cho22b, Proposition A.3, Proposition A.4, (A.0.4), (A.0.5)]. \square

Proposition 8.6. For $n-h \leq t \leq n-1$, we have

$$\frac{\beta_t^{n-h} W_{n-h,t}(B, 1)}{W_{n,n}(A_n, 1)} = \frac{-(-q)^{\frac{(n-t)(n-t-1-2h)}{2}}}{1 - (-q)^{-(n-t)}} \frac{\text{Den}(I_{n,t-n+h}, T)}{\text{Den}(I_{n,t-n+h}, I_{n,t-n+h})}.$$

Proof. By Definition 7.5, we have that

$$\begin{aligned} \beta_t^{n-h} &= (-q)^{-(n-t)(n+h)} \frac{(-1)^{n-1} (-q)^{\frac{n(n+1)}{2} - (n-t)} \prod_{l=n+1-t}^n (1 - (-q)^{-l}) \prod_{l=1}^{n-t-1} (1 - (-q)^{-l}) \prod_{l=1}^n (1 - (-q)^{-l})}{(-1)^t (-q)^{2n(n-t) + \frac{t(t+1)}{2}} \prod_{l=1}^t (1 - (-q)^{-l}) \prod_{l=1}^{2n-t} (1 - (-q)^{-l})} \\ &= (-1)^{n-t-1} (-q)^{-\frac{(n-t)(1+2h+5n-t)}{2}} \frac{\prod_{l=n+1-t}^n (1 - (-q)^{-l}) \prod_{l=1}^{n-t-1} (1 - (-q)^{-l}) \prod_{l=1}^n (1 - (-q)^{-l})}{\prod_{l=1}^t (1 - (-q)^{-l}) \prod_{l=1}^{2n-t} (1 - (-q)^{-l})}. \end{aligned}$$

Also, by [Cho22b, Proposition A.4], we have that

$$\text{Den}(\pi A_t, I_{n-h}) = \text{Den}\left(\begin{pmatrix} \pi I_{2n-t} & \\ & I_t \end{pmatrix}, I_{n-h} \right) = \prod_{l=t-n+h+1}^t (1 - (-q)^{-l}),$$

$$\text{Den}(I_{n+h,t-n+h}, I_h) = \prod_{l=2n-h-t+1}^{2n-t} (1 - (-q)^{-l}),$$

$$\text{Den}(I_{n,t-n+h}, I_{n,t-n+h}) = q^{(t-n+h)^2} \prod_{l=1}^{2n-t-h} (1 - (-q)^{-l}) \prod_{l=1}^{t-n+h} (1 - (-q)^{-l}),$$

and

$$W_{n,n}(A_n, 1) = q^{-3n^2} \prod_{l=1}^n (1 - (-q)^{-l})^2.$$

Combining these and Proposition 8.5, we get the proposition. \square

Combining Proposition 8.1 and Proposition 8.6, we get the following corollary which compares the analytic sides of Conjecture 3.3 and Conjecture 7.6.

Corollary 8.7. *Assume that B is of the in (7.1). By Proposition 8.1, Proposition 8.5, Proposition 8.6, and [Cho22b, Proposition A.4], we can write $\frac{1}{W_{n,n}(A_n, 1)} \{W'_{n-h,n}(B) - \sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1)\}$ in terms of usual representation densities as follows:*

$$\begin{aligned} & \frac{1}{W_{n,n}(A_n, 1)} \{W'_{n-h,n}(B) - \sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1)\} \\ &= \frac{\text{Den}'(I_{n,h}, T)}{\text{Den}(I_{n,h}, I_{n,h})} + \sum_{k=0}^{h-1} \frac{(-q)^{-\frac{(h-k)(h+k+1)}{2}}}{1 - (-q)^{-(h-k)}} \frac{\text{Den}(I_{n,k}, T)}{\text{Den}(I_{n,k}, I_{n,k})}. \end{aligned}$$

By Corollary 8.7, Conjecture 7.7 can be rewritten as follows.

Conjecture 8.8 (\mathcal{Z} -cycles in $\mathcal{N}_n^{[h]}$). *For a basis $\{x_1, \dots, x_n\}$ of \mathbb{V} , and special cycles $\mathcal{Z}(x_1), \dots, \mathcal{Z}(x_n)$ in $\mathcal{N}_n^{[h]}$, we have*

$$\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_n)}) = \frac{\text{Den}'(I_{n,h}, T)}{\text{Den}(I_{n,h}, I_{n,h})} + \sum_{k=0}^{h-1} \frac{(-q)^{-\frac{(h-k)(h+k+1)}{2}}}{1 - (-q)^{-(h-k)}} \frac{\text{Den}(I_{n,k}, T)}{\text{Den}(I_{n,k}, I_{n,k})}.$$

Here, χ is the Euler-Poincaré characteristic and $\otimes^{\mathbb{L}}$ is the derived tensor product. Also, T is the matrix

$$T = \left((x_i, x_j) \right)_{1 \leq i, j \leq n}.$$

Proposition 8.9. *Conjecture 3.3 is equivalent to Conjecture 8.8.*

Proof. The intersection numbers from both conjectures are by definition the same. Hence we only need to show the analytic sides of both conjectures agree. By Corollary 8.7, we only need to show β_t^{n-h} from Conjecture 7.6 is the same as $c_{n,t}$ from Conjecture 3.3. Since the $c_{n,t}$ is characterized by $\partial \text{Pden}_{n,h}(I_{n,t}) = 0$ for $t \leq h-1$ and $t \equiv h+1 \pmod{2}$, we only need to show $D_{n,h}(I_{n,t}) = 0$ for $t \leq h-1$ and $t \equiv h+1 \pmod{2}$. This is proved in Proposition 9.6 via a method we used throughout §9 so we postpone the proof. \square

Remark 8.10. Before we start to find the constants $D_{n,h}(\lambda)$, let us provide a brief explanation of the forthcoming steps. By Proposition 8.1, we know that $\frac{W'_{n-h,n}(B)}{W_{n,n}(A_n, 1)}$ is a constant multiple of $\text{Den}'(I_{n,h}, T)$. Also, we know how to write $\text{Den}'(I_n, T)$ in terms of a linear sum of representation densities by [CY20] and [LZ22a, Theorem 3.5.1] (see Proposition 8.15 below).

Now, we consider $\text{Den}'(I_{n,h}, T) - (-q)^{nh} \text{Den}'(I_n, T)$. Then, it is possible to write the difference between these two in terms of a certain linear sum of representation densities (see (8.11)). Therefore, we can use the linear sum expression of $(-q)^{nh} \text{Den}'(I_n, T)$ and the difference $\text{Den}'(I_{n,h}, T) - (-q)^{nh} \text{Den}'(I_n, T)$ to find all constants $D_{n,h}(\lambda)$. This is what we will do in the next few pages.

Definition 8.11. (1) For $Y \in \mathcal{R}_n$, we define

$$t_0(Y) = |\{e_i \mid e_i \geq 0\}|,$$

and for $k \geq 1$, we define

$$t_k(Y) = |\{e_i \mid e_i = -k\}|.$$

(2) For $\eta \in \mathcal{R}_n^{0+}$, and $k \geq 0$, we define

$$t_k(\eta) = |\{\eta_i \mid \eta_i = k\}|,$$

and

$$t_{\geq k}(\eta) = |\{\eta_i \mid \eta_i \geq k\}|.$$

(3) For $Y \in \mathcal{R}_n$, we define

$$t(Y) = (t_0(Y), t_1(Y), \dots).$$

Similarly, for $\eta \in \mathcal{R}_n^{0+}$, we define

$$t(\eta) = (t_0(\eta), t_1(\eta), \dots).$$

Definition 8.12. (1) For $\lambda = (\bar{\lambda}, \overbrace{1, \dots, 1}^{t_1(\lambda)}, \overbrace{0, \dots, 0}^{t_0(\lambda)}) \in \mathcal{R}_n^{0+}$, and $0 \leq s \leq t_0(\lambda)$, we define

$$\lambda_s^+ := (\bar{\lambda}, \overbrace{1, \dots, 1}^{t_1(\lambda)+s}, \overbrace{0, \dots, 0}^{t_0(\lambda)-s}),$$

by replacing s zeros by s 1's.

(2) For $\lambda = (\bar{\lambda}, \overbrace{1, \dots, 1}^{t_1(\lambda)}, \overbrace{0, \dots, 0}^{t_0(\lambda)}) \in \mathcal{R}_n^{0+}$ and $0 \leq s \leq t_1(\lambda)$, we define

$$\lambda_s^- = (\bar{\lambda}, \overbrace{1, \dots, 1}^{t_1(\lambda)-s}, \overbrace{0, \dots, 0}^{t_0(\lambda)+s}),$$

by replacing s 1's by s zeros.

(3) For $0 \leq l \leq n$ and $\lambda \in \mathcal{R}_n^{0+}$ such that $t_0(\lambda) \geq l$, we define $\lambda^{\vee l}$ as the element in \mathcal{R}_{n-l}^{0+} such that $\lambda = (\lambda^{\vee l}, 0, \dots, 0)$.

Definition 8.13. Assume that $k \geq 0$, $\alpha, \eta \in \mathcal{R}_n^{0+}$, $Y \in \mathcal{R}_n^{0+}$, and $t(\eta) = t(Y)$.

(1) We define

$$\mathcal{B}_k(Y) = \sum_i \min(0, e_i + k) - \min(0, e_i),$$

and

$$\mathcal{B}_k(\eta) = \sum_i \min(k, \eta_i).$$

Note that $\mathcal{B}_k(\eta) = \mathcal{B}_k(Y)$ since $t(\eta) = t(Y)$.

(2) We define

$$\mathcal{B}_\alpha(Y) = \sum_i \mathcal{B}_{\alpha_i}(Y),$$

and

$$\mathcal{B}_\alpha(\eta) = \sum_i \mathcal{B}_{\alpha_i}(\eta).$$

Note that $\mathcal{B}_\alpha(\eta) = \mathcal{B}_\alpha(Y)$ since $t(\eta) = t(Y)$.

(3) We define $f(Y)$ by

$$f(Y) = \prod_i (-q)^{n \min(0, e_i)}.$$

By Lemma 7.3 and the fact that \mathcal{R}_n forms a complete set of representatives of $\Gamma_n \backslash X_n(F)$, we have that

$$(8.1) \quad \text{Den}'(I_{n,h}, B) = \sum_{Y \in \mathcal{R}_n} \frac{\mathcal{G}(Y, B) \mathcal{F}'(Y, I_{n,h})}{\alpha(Y; \Gamma_n)},$$

$$(8.2) \quad \text{Den}'(I_n, B) = \sum_{Y \in \mathcal{R}_n} \frac{\mathcal{G}(Y, B) \mathcal{F}'(Y, I_n)}{\alpha(Y; \Gamma_n)},$$

and for $\lambda \in \mathcal{R}_n^{0+}$,

$$(8.3) \quad \text{Den}(A_\lambda, B) = \sum_{Y \in \mathcal{R}_n} \frac{\mathcal{G}(Y, B) \mathcal{F}(Y, A_\lambda)}{\alpha(Y; \Gamma_n)}.$$

By [Cho23, Lemma 3.2, Section 3.2], we know that $\mathcal{F}'(Y, I_{n,h})$ can be written uniquely as a linear sum of $\mathcal{F}(Y, A_\lambda)$, $\lambda \in \mathcal{R}_n^{0+}$. As in the proof of [Cho22b, Lemma 3.15], we can compute that for

$$Y = Y_{\sigma, e} = \sigma \begin{pmatrix} \pi^{e_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{e_n} \end{pmatrix},$$

we have

$$\mathcal{F}'(Y, I_{n,h}) = \sum_j \min(0, e_j) (-q)^{h \mathcal{B}_1(Y)} f(Y),$$

$$\mathcal{F}'(Y, I_n) = \sum_j \min(0, e_j) f(Y),$$

and

$$\mathcal{F}(Y, A_\lambda) = (-q)^{\mathcal{B}_\lambda(Y)} f(Y).$$

Therefore, we have that (cf. [Cho23, (4.1.1)])

(8.4)

$$\mathcal{F}'(Y, I_{n,h}) - (-q)^{hn} \mathcal{F}'(Y, I_n) = \begin{cases} 0 & \text{if } t_0(Y) = 0, \\ \left(\sum_j \min(0, e_j) \right) f(Y) \left((-q)^{h(n-1)} - (-q)^{hn} \right) & \text{if } t_0(Y) = 1, \\ \vdots & \\ \left(\sum_j \min(0, e_j) \right) f(Y) \left((-q)^{h(n-k)} - (-q)^{hn} \right) & \text{if } t_0(Y) = k, \\ \vdots & \\ \left(\sum_j \min(0, e_j) \right) f(Y) \left(1 - (-q)^{hn} \right) & \text{if } t_0(Y) = n. \end{cases}$$

Now, let us define the following constants and matrices.

Definition 8.14.

(1) (cf. [Cho23, Lemma 4.4]) For $0 \leq i \leq l$, we define constants d_{il} by

$$d_{il} = (-q)^{-in} \prod_{\substack{0 \leq m \leq l \\ m \neq i}} \frac{1}{((-q)^{-i} - (-q)^{-m}}.$$

Therefore, we have

$$\frac{d_{il}}{d_{i+1,l}} = -(-q)^{n+i+1-l} \frac{(1 - (-q)^{-(i+1)})}{(1 - (-q)^{-(l-i)}}.$$

(2) We define the upper triangular $(n+1) \times (n+1)$ matrix Δ by

$$\Delta = \begin{pmatrix} d_{00} & d_{01} & d_{02} & \dots & d_{0n} \\ 0 & d_{11} & d_{12} & \dots & d_{1n} \\ 0 & 0 & d_{22} & \dots & d_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{pmatrix}.$$

(3) For $l \leq 0$, we define \mathfrak{M}_l by

$$\mathfrak{M}_l : \begin{pmatrix} 1 & (-q)^{2n} & \dots & (-q)^{2ln} \\ 1 & (-q)^{2n-1} & \dots & (-q)^{l(2n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (-q)^{2n-l} & \dots & (-q)^{l(2n-l)} \end{pmatrix}.$$

(4) For $0 \leq i, j \leq n$, we define constants \mathcal{A}_{ij} by

$$(\mathcal{A}_{ij})_{0 \leq i, j \leq n} = \mathfrak{M}_n \Delta.$$

(5) For $0 \leq i \leq n$, we define constants \mathcal{K}_i by

$$(\mathcal{A}_{ij}) \begin{pmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \vdots \\ \mathcal{K}_n \end{pmatrix} = \begin{pmatrix} 0 \\ (-q)^{h(n-1)} - (-q)^{hn} \\ \vdots \\ 1 - (-q)^{hn} \end{pmatrix}.$$

Proposition 8.15. [Cho23, Proposition 3.3, Proposition 3.4] *We define the following constants.*

(1) *For $\alpha \in \mathcal{R}_n^{0+}$ such that $\sum \alpha_i = \text{odd}$, we write \mathcal{C}_α for*

$$\mathcal{C}_\alpha = \prod_{i=1}^{t_{\geq 1}(\alpha)-1} (1 - (-q)^i).$$

Here, we define $\mathcal{C}_\alpha = 1$ if $t_{\geq 1}(\alpha) = 1$.

(2) *For $\alpha \in \mathcal{R}_n^{0+}$ such that $\sum \alpha_i = \text{even}$ and $\alpha \neq (0, 0, \dots, 0)$, we write \mathcal{C}_α for*

$$\mathcal{C}_\alpha = - \prod_{i=1}^{t_{\geq 1}(\alpha)-1} (1 - (-q)^i).$$

Here, we define $\mathcal{C}_\alpha = -1$ if $t_{\geq 1}(\alpha) = 1$.

Also, if $\alpha = (0, 0, \dots, 0)$, we define

$$\mathcal{C}_\alpha = \frac{\text{Den}'(I_n, I_n)}{\text{Den}(I_n, I_n)}.$$

Then, we have that

$$\frac{\text{Den}'(I_n, B)}{\text{Den}(I_n, I_n)} = \sum_{\alpha \in \mathcal{R}_n^{0+}} \mathcal{C}_\alpha \frac{\text{Den}(A_\alpha, B)}{\text{Den}(A_\alpha, A_\alpha)}.$$

Now, by [Cho23, (4.1.5)], we have that for $\lambda \in \mathcal{R}_n^{0+}$ with $t_0(Y) \geq l$,

$$(8.5) \quad \sum_{0 \leq i \leq l} d_{il} \mathcal{F}(Y, A_{\lambda_i^+}) = \begin{cases} 0 = A_{0l} \mathcal{F}(Y, A_\lambda) & \text{if } t_0(Y) = 0, \\ \vdots & \vdots \\ 0 = A_{l-1,l} \mathcal{F}(Y, A_\lambda) & \text{if } t_0(Y) = l-1, \\ (-q)^{\mathcal{B}_\lambda(Y)} f(Y) = \mathcal{F}(Y, A_\lambda) = A_{ll} \mathcal{F}(Y, A_\lambda) & \text{if } t_0(Y) = l, \\ A_{kl} \mathcal{F}(Y, A_\lambda) & \text{if } t_0(Y) = k, \ l+1 \leq k \leq n. \end{cases}$$

Also, by Proposition 8.15, we have that for $Y \in \mathcal{R}_{n-l}^{0+}$,

$$\begin{aligned} \frac{\mathcal{F}'(Y, I_{n-l})}{\text{Den}(I_{n-l}, I_{n-l})} &= \sum_{\bar{\lambda} \in \mathcal{R}_{n-l}^{0+}} \mathcal{C}_{\bar{\lambda}} \frac{\mathcal{F}(Y, A_{\bar{\lambda}})}{\text{Den}(A_{\bar{\lambda}}, A_{\bar{\lambda}})} \\ \iff \sum_i \min(0, e_i) f(Y) &= \sum_{\bar{\lambda} \in \mathcal{R}_{n-l}^{0+}} \mathcal{C}_{\bar{\lambda}} \frac{\text{Den}(I_{n-l}, I_{n-l})}{\text{Den}(A_{\bar{\lambda}}, A_{\bar{\lambda}})} (-q)^{\mathcal{B}_{\bar{\lambda}}(Y)} f(Y) \\ \iff \sum_i \min(0, e_i) &= \sum_{\bar{\lambda} \in \mathcal{R}_{n-l}^{0+}} \mathcal{C}_{\bar{\lambda}} \frac{\text{Den}(I_{n-l}, I_{n-l})}{\text{Den}(A_{\bar{\lambda}}, A_{\bar{\lambda}})} (-q)^{\mathcal{B}_{\bar{\lambda}}(Y)}. \end{aligned}$$

Since running $\lambda = (\lambda^{\vee l}, 0, \dots, 0)$ over \mathcal{R}_n^{0+} with $t_0(Y) \geq l$ is equivalent to running $\lambda^{\vee l}$ over \mathcal{R}_{n-l}^{0+} by removing l zeros, we have that

$$(8.6) \quad \sum_{\lambda \in \mathcal{R}_n^{0+}, t_0(Y) \geq l} \mathcal{C}_{\lambda^{\vee l}} \frac{\text{Den}(I_{n-l}, I_{n-l})}{\text{Den}(A_{\lambda^{\vee l}}, A_{\lambda^{\vee l}})} \sum_{0 \leq i \leq l} d_{il} \mathcal{F}_0(Y, A_{\lambda_i^+})$$

$$= \begin{cases} 0 = \mathcal{A}_l \sum_j \min(0, e_j) f(Y) & \text{if } t_0(Y) \leq l-1 \\ \mathcal{A}_l \sum_j \min(0, e_j) f(Y) & \text{if } t_0(Y) \geq l. \end{cases}$$

In Definition 8.14 (5), we defined constants \mathcal{K}_i such that

$$(\mathcal{A}_{ij}) \begin{pmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \vdots \\ \mathcal{K}_n \end{pmatrix} = \begin{pmatrix} 0 \\ (-q)^{h(n-1)} - (-q)^{hn} \\ \vdots \\ 1 - (-q)^{hn} \end{pmatrix}.$$

Therefore, we have that

$$\begin{aligned} & \sum_{l=0}^n \mathcal{K}_l \left\{ \sum_{\lambda \in \mathcal{R}_n^{0+}, t_0(Y) \geq l} \mathcal{C}_{\lambda^{\vee l}} \frac{\text{Den}(I_{n-l}, I_{n-l})}{\text{Den}(A_{\lambda^{\vee l}}, A_{\lambda^{\vee l}})} \sum_{0 \leq i \leq l} d_{il} \mathcal{F}_0(Y, A_{\lambda_i^+}) \right\} \\ &= \begin{cases} 0 & \text{if } t_0(Y) = 0, \\ \left(\sum_j \min(0, e_j) \right) f(Y) ((-q)^{h(n-1)} - (-q)^{hn}) & \text{if } t_0(Y) = 1, \\ \vdots \\ \left(\sum_j \min(0, e_j) \right) f(Y) ((-q)^{h(n-k)} - (-q)^{hn}) & \text{if } t_0(Y) = k, \\ \vdots \\ \left(\sum_j \min(0, e_j) \right) f(Y) (1 - (-q)^{hn}) & \text{if } t_0(Y) = n. \end{cases} \end{aligned}$$

By comparing this with (8.4), we have that

$$(8.7) \quad \begin{aligned} & \mathcal{F}'(Y, I_{n,h}) - (-q)^{hn} \mathcal{F}'(Y, I_n) \\ &= \sum_{l=0}^n \mathcal{K}_l \left\{ \sum_{\lambda \in \mathcal{R}_n^{0+}, t_0(Y) \geq l} \mathcal{C}_{\lambda^{\vee l}} \frac{\text{Den}(I_{n-l}, I_{n-l})}{\text{Den}(A_{\lambda^{\vee l}}, A_{\lambda^{\vee l}})} \sum_{0 \leq i \leq l} d_{il} \mathcal{F}_0(Y, A_{\lambda_i^+}) \right\}. \end{aligned}$$

Now, we have the following lemma.

Lemma 8.16. (cf. [Cho23, Lemma 4.5]) For $i = 0$ and $h + 1 \leq i \leq n$, $\mathcal{K}_i = 0$. Also, $d_{hh}\mathcal{K}_h = 1$ and for $1 \leq l \leq h - 1$, we have

$$d_{h-l, h-l} \mathcal{K}_{h-l} = (-q)^n \frac{1 - (-q)^{-h+l-1}}{1 - (-q)^{-l}} d_{h-l+1, h-l+1} \mathcal{K}_{h-l+1}.$$

Proof. As in [Cho23, Lemma 4.3], one can show that

$$(8.8) \quad \Delta \begin{pmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \cdots \\ \mathcal{K}_n \\ \cdots \\ \mathcal{K}_{2n} \end{pmatrix} = \begin{pmatrix} -(-q)^{nh} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \text{1st entry} \\ \\ \\ \text{(h+1)-th entry} \\ \\ \end{array}.$$

Since Δ is an upper triangular matrix, we have that $\mathcal{K}_i = 0$ for $h + 1 \leq i \leq n$. Also, the $(h + 1)$ -th row of (8.8) implies that

$$d_{hh}\mathcal{K}_h + d_{h,h+1}\mathcal{K}_{h+1} + \cdots + d_{h,n}\mathcal{K}_n = 1,$$

and hence $d_{hh}\mathcal{K}_h = 1$. Now, the proof of the last statement is almost identical to the proof of [Cho23, Lemma 4.5]. \square

Now, we are ready to write the following proposition on the constants $D_{n,h}(\lambda)$.

Proposition 8.17. (cf. [Cho23, Theorem 4.7]) For $0 \leq h \leq n$ and $\lambda \in \mathcal{R}_n^{0+}$ such that $\lambda \neq (1^t, 0^{n-t})$, $t \leq h-1$, we have

$$(8.9) \quad D_{n,h}(\lambda) = \frac{q^{h(n-h)} \prod_{l=1}^n (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l}) \prod_{l=1}^{n-h} (1 - (-q)^{-l})} \mathcal{C}_\lambda \\ + \sum_{\substack{1 \leq i \leq h \\ \max\{i-t_0(\lambda), 0\} \leq s \\ \leq \min\{i, t_1(\lambda)\}}} (-q)^{n(h-i)+(i-s)(2n-i+s+1)/2-h^2+s(2n-2t_0(\lambda)-s)} (-1)^{i+h} \\ \times \frac{\prod_{l=1}^{n-i} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h} (1 - (-q)^{-l}) \prod_{l=1}^h (1 - (-q)^{-l})} \times \frac{\prod_{l=s+1}^h (1 - (-q)^{-l})}{\prod_{l=1}^{h-i} (1 - (-q)^{-l}) \prod_{l=1}^{i-s} (1 - (-q)^{-l})} \\ \times \frac{\prod_{l=1}^{l=t_0(\lambda)} (1 - (-q)^{-l}) \prod_{l=1}^{l=t_1(\lambda)} (1 - (-q)^{-l})}{\prod_{l=1}^{l=t_0(\lambda)-i+s} (1 - (-q)^{-l}) \prod_{l=1}^{l=t_1(\lambda)-s} (1 - (-q)^{-l})} \times \mathcal{C}_{(\lambda_s^-)^{\vee_i}}.$$

Here, we choose the following convention: for $k \leq 0$, we assume that

$$\prod_{l=1}^k (*) = 1 \quad \text{and} \quad \sum_{l=1}^k (*) = 0.$$

If $\lambda = (1^t, 0^{n-t})$, $t \leq h-1$, then $D_{n,h}(\lambda) =$ the right hand side of (8.9) $+ \frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}}$. In particular, $D_{n,h}(\lambda)$ depends only on n , $t_0(\lambda)$, $t_1(\lambda)$, and the parity of $\sum_i \lambda_i$.

Remark 8.18. Note that when $h = 0$, the sum $\sum_{\substack{1 \leq i \leq h \\ \max\{i-t_0(\lambda), 0\} \leq s \leq \min\{i, t_1(\lambda)\}} (*)$ in (8.9) is empty, and hence $D_{n,0}(\lambda) = \mathcal{C}_\lambda$ which is obvious by construction. Therefore, Proposition 8.17 decomposes $D_{n,h}$ into a weighted summation of $D_{n,0}$, which is the Cho-Yamauchi constant in the good reduction case.

Proof of Proposition 8.17. The proof is almost identical to the proof of [Cho23, Theorem 4.7]; therefore, let me just write which parts are different. First, note that for $B \in X_n(F)$ of the form in (7.1), Proposition 8.1 implies that

$$(8.10) \quad \frac{W'_{n-h,n}(B)}{W_{n,n}(A_n, 1)} = \frac{q^{-4n^2+(n+h)(n-h)} \text{Den}(\pi A_n, I_{n-h}) \text{Den}(I_{n+h,h}, I_h) \text{Den}'(I_{n,h}, T)}{W_{n,n}(A_n, 1)}.$$

Now, by (8.1), (8.2), (8.3), and (8.7), we have

$$(8.11) \quad \text{Den}'(I_{n,h}, T) = (-q)^{nh} \text{Den}'(I_n, T) + \sum_{l=0}^n \mathcal{K}_l \left\{ \sum_{\lambda \in \mathcal{R}_n^{0+}, t_0(Y) \geq l} \mathcal{C}_{\lambda^{\vee_l}} \frac{\text{Den}(I_{n-l}, I_{n-l})}{\text{Den}(A_{\lambda^{\vee_l}}, A_{\lambda^{\vee_l}})} \sum_{0 \leq i \leq l} d_{il} \text{Den}(A_{\lambda_i^+}, T) \right\}.$$

Now, we can easily follow the proof of [Cho23, Theorem 4.7] by using Lemma 8.16, (8.10), and (8.11). Also, for $\lambda = (1^t, 0^{n-t})$, $t \leq h-1$, we use Proposition 8.6. \square

Note that $D_{n,h}(\lambda)$ depends only on $t_{\geq 2}(\lambda)$, $t_1(\lambda)$, $t_0(\lambda)$, and the parity of $\sum_i \lambda_i$. Also, by Remark 8.4, we only need to consider λ such that $\sum_i \lambda_i \equiv h+1 \pmod{2}$. Also, later on, we will establish inductive formulas relating $D_{n,h}(\lambda)$ for λ with different $t_{\geq 2}(\lambda)$, $t_1(\lambda)$ and $t_0(\lambda)$. So we introduce the following notation to streamline the computation.

Definition 8.19. Assume that $h, n, a, b, c, i, j, s \geq 0$, $0 \leq s \leq i \leq h \leq n$, $0 \leq j \leq n$, and $n = a + b + c$.

(1) If $c \neq n$, we write $\mathcal{C}_j(a, b, c)$ for

$$\mathcal{C}_j(a, b, c) = (-1)^{j+1} \prod_{i=1}^{a+b-1} (1 - (-q)^i).$$

Also, if $c = n$, we define

$$\mathcal{C}_j(0, 0, n) = \frac{\text{Den}'(I_n, I_n)}{\text{Den}(I_n, I_n)}.$$

Then, for $\lambda \in \mathcal{R}_n^{0+}$, we have that

$$\mathcal{C}_\lambda = \mathcal{C}_{|\lambda|}(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)).$$

(2) Let $M_{n,h}(a, b, c, i, s)$ be the constant

$$\begin{aligned} M_{n,h}(a, b, c, i, s) &= (-q)^{n(h-i)+(i-s)(2n-i+s+1)/2-h^2+s(2n-2c-s)} (-1)^{i+h} \\ &\times \frac{\prod_{l=1}^{n-i} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h} (1 - (-q)^{-l}) \prod_{l=1}^h (1 - (-q)^{-l})} \times \frac{\prod_{l=s+1}^h ((1 - (-q)^{-l})}{\prod_{l=1}^{h-i} (1 - (-q)^{-l}) \prod_{l=1}^{i-s} (1 - (-q)^{-l})} \\ &\times \frac{\prod_{l=1}^c (1 - (-q)^{-l}) \prod_{l=1}^b (1 - (-q)^{-l})}{\prod_{l=1}^{c-i+s} (1 - (-q)^{-l}) \prod_{l=1}^{b-s} (1 - (-q)^{-l})} \times \mathcal{C}_{h+1-s}(a, b-s, c+s-i). \end{aligned}$$

(3) For $(a, b, c) \neq (0, t, n-t)$, $t \leq h-1$, we write $D_{n,h}(a, b, c)$ for

$$D_{n,h}(a, b, c) = \sum_{\substack{0 \leq i \leq h \\ \max(i-c, 0) \leq s \leq \min(i, b)}} M_{n,h}(a, b, c, i, s) = \sum_{\substack{0 \leq s \leq \min(h, b) \\ s \leq i \leq \min(s+c, h)}} M_{n,h}(a, b, c, i, s).$$

(4) For $(a, b, c) = (0, t, n-t)$, $t \leq h-1$, $t \equiv h+1 \pmod{2}$, we write $D_{n,h}(a, b, c)$ for

$$\begin{aligned} D_{n,h}(0, t, n-t) &= \sum_{\substack{0 \leq i \leq h \\ \max(i-c, 0) \leq s \leq \min(i, b)}} M_{n,h}(a, b, c, i, s) + \frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}} \\ &= \sum_{\substack{0 \leq s \leq \min(h, b) \\ s \leq i \leq \min(s+c, h)}} M_{n,h}(a, b, c, i, s) + \frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}}. \end{aligned}$$

Proposition 8.20. For $\lambda \in \mathcal{R}_n^{0+}$, we have that

$$D_{n,h}(\lambda) = D_{n,h}(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)).$$

Proof. This follows from the definitions of $M_{n,h}(a, b, c, i, s)$, $D_{n,h}(a, b, c)$, Proposition 8.17, and the fact that

$$M_{n,h}(a, b, c, 0, 0) = \frac{q^{h(n-h)} \prod_{l=1}^n (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l}) \prod_{l=1}^{n-h} (1 - (-q)^{-l})} \mathcal{C}_{h+1}(a, b, c).$$

□

9. INDUCTIVE RELATIONS AMONG CHO-YAMAUCHI CONSTANTS

In this section, we will prove some inductive relations among $D_{n,h}(\lambda)$. The main result of this section can be summarized as follows: Let $0 \leq a, b, c, h, \leq n$, $a + b + c = n$. Assume further that $(a - 1, b + 1, c) \neq (0, h, n - h)$. Then, we have the following result.

(1) (Theorem 9.4) If $c \leq n - h$ and $a \geq 1$, then

$$D_{n,h}(a, b, c) - D_{n,h}(a - 1, b + 1, c) = -(-q)^{2n-h-1-b-2c} D_{n-1,h-1}(a - 1, b, c).$$

(2) (Lemma 9.5, Proposition 9.6) If $c > n - h$, then $D_{n,h}(a, b, c) = 0$.

(3) (Theorem 9.8 (1)) Assume that $a = 0$ and $h + 1 \leq b$. Then, we have

$$D_{n,h}(0, b, c) = \frac{\prod_{l=h+1}^b (1 - (-q)^l)}{(1 - (-q)^{b-h})}.$$

(4) (Theorem 9.8 (2)) Assume that $a = 1$ and $h - 1 \leq b$. Then, we have

$$D_{n,h}(1, b, c) = \begin{cases} 1 & \text{if } b = h - 1, h; \\ \prod_{l=h+1}^b (1 - (-q)^l) & \text{if } b \geq h + 1. \end{cases}$$

Here, the main reason for excluding the case $(a - 1, b + 1, c) = (0, h, n - h)$ is because it is not necessary by Remark 8.4.

First, let us prove the following lemma.

Lemma 9.1. *Let $0 \leq a, b, c, h \leq n$, $a + b + c = n$, and $0 \leq s \leq \min(h, b)$.*

(1) *If $c \leq n - h$ and $(a, b, c) \neq (0, h, n - h)$, we have that*

$$\begin{aligned} & \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) \\ &= (-1)^{n+c+s+1} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch + \frac{s}{2} - \frac{s^2}{2}} \\ & \quad \times \frac{\prod_{l=1}^b (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l})}{\prod_{l=1}^{b-s} (1 - (-q)^{-l}) \prod_{l=1}^s (1 - (-q)^{-l}) \prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})}. \end{aligned}$$

(2) *If $c > n - h$ and $a \neq 0$, we have that $\sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) = 0$.*

Proof. By our assumption, we have that $(a, b - s, c + s - i) \neq (0, 0, n - i)$ for any $i \leq s \leq \min(s + c, h)$. Therefore, by Definition 8.19 (1), we have that

$$\begin{aligned} \mathcal{C}_{h+1-s}(a, b - s, c + s - i) &= (-1)^{h-s} \prod_{l=1}^{n-c-s-1} (1 - (-q)^l) \\ &= (-1)^{n+c+h+1} (-q)^{(n-c-s-1)(n-c-s)/2} \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l}), \end{aligned}$$

which is independent of i . Also, by Definition 8.19 (2), we have that

$$M_{n,h}(a, b, c, i, s) = C_{h+1-s}(a, b-s, c+s-i)Y_{n,h}(b, c, s)X_{n,h}(c, i, s),$$

where

$$Y_{n,h}(b, c, s) := (-1)^h(-q)^{-h^2+hn-\frac{s}{2}-\frac{3s^2}{2}-2cs+ns} \frac{\prod_{l=1}^c(1-(-q)^{-l})\prod_{l=1}^b(1-(-q)^{-l})}{\prod_{l=1}^{n-h}(1-(-q)^{-l})\prod_{l=1}^s(1-(-q)^{-l})\prod_{l=1}^{b-s}(1-(-q)^{-l})},$$

which is independent of i , and

$$X_{n,h}(c, i, s) := \frac{(-1)^i(-q)^{\frac{i}{2}-\frac{i^2}{2}+is}\prod_{l=1}^{n-i}(1-(-q)^{-l})}{\prod_{l=1}^{h-i}(1-(-q)^{-l})\prod_{l=1}^{i-s}(1-(-q)^{-l})\prod_{l=1}^{c-i+s}(1-(-q)^{-l})}.$$

Now, we note that Lemma 9.1 follows from the following claim

$$(9.1) \quad \sum_{i=s}^{\min(s+c,h)} X_{n,h}(c, i, s) = \begin{cases} \frac{(-1)^s(-q)^{-ch+cs+\frac{s}{2}+\frac{s^2}{2}}\prod_{l=1}^{n-h}(1-(-q)^{-l})\prod_{l=1}^{n-s-c}(1-(-q)^{-l})}{\prod_{l=1}^{h-s}(1-(-q)^{-l})\prod_{l=1}^c(1-(-q)^{-l})\prod_{l=1}^{n-h-c}(1-(-q)^{-l})} & \text{if } c \leq n-h, \\ 0 & \text{if } c > n-h. \end{cases}$$

Hence it suffices to show that (9.1) holds.

If $h = s$, then

$$\sum_{i=s}^{\min(s+c,h)} X_{n,h}(c, i, s) = X_{n,h}(c, s, s) = \frac{(-1)^s(-q)^{\frac{s}{2}+\frac{s^2}{2}}\prod_{l=1}^{n-s}(1-(-q)^{-l})}{\prod_{l=1}^c(1-(-q)^{-l})}.$$

Therefore, (9.1) holds in this case.

Similarly, if $c = 0$, then

$$\sum_{i=s}^{\min(s+c,h)} X_{n,h}(c, i, s) = X_{n,h}(c, s, s) = \frac{(-1)^s(-q)^{\frac{s}{2}+\frac{s^2}{2}}\prod_{l=1}^{n-s}(1-(-q)^{-l})}{\prod_{l=1}^{h-s}(1-(-q)^{-l})}.$$

Therefore, (9.1) holds in this case.

Now, assume that $c \geq 1$ and $h > s$. In Lemma 9.2 below, we will prove that

$$(9.2) \quad \sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} = \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \sum_{i=s}^{\min(s+c-1,h)} \frac{X_{n-1,h}(c-1, i, s)}{X_{n-1,h}(c-1, s, s)}.$$

Assume (9.2) holds for now. Then we are ready to prove (9.1).

If $c \leq n - h$, by applying (9.2) repeatedly, we have that

$$\begin{aligned} \sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} &= (-q)^{-c(h-s)} \frac{\prod_{l=n-h-c+1}^{n-h} (1 - (-q)^{-l})}{\prod_{l=n-s-c+1}^{n-s} (1 - (-q)^{-l})} \sum_{i=s}^{\min(s+0,h)} \frac{X_{n-c,h}(0, i, s)}{X_{n-c,h}(0, s, s)} \\ &= (-q)^{-c(h-s)} \frac{\prod_{l=n-h-c+1}^{n-h} (1 - (-q)^{-l})}{\prod_{l=n-s-c+1}^{n-s} (1 - (-q)^{-l})}. \end{aligned}$$

Combining this with

$$X_{n,h}(c, s, s) = \frac{(-1)^s (-q)^{\frac{s}{2} + \frac{s^2}{2}} \prod_{l=1}^{n-s} (1 - (-q)^{-l})}{\prod_{l=1}^{h-s} (1 - (-q)^{-l}) \prod_{l=1}^c (1 - (-q)^{-l})},$$

we see that (9.1) holds when $c \leq n - h$.

Now, assume that $c > n - h$. By (9.2), we have that

$$\sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} = (-q)^{-(n-h)(h-s)} \frac{\prod_{l=1}^{n-h} (1 - (-q)^{-l})}{\prod_{l=h-s+1}^{n-s} (1 - (-q)^{-l})} \sum_{i=s}^{\min(s+c-n+h,h)} \frac{X_{h,h}(c-n+h, i, s)}{X_{h,h}(c-n+h, s, s)}.$$

Therefore, it suffices to show that $\sum_{i=s}^{\min(s+c-n+h,h)} \frac{X_{h,h}(c-n+h, i, s)}{X_{h,h}(c-n+h, s, s)} = 0$.

Note that $s \leq \min(h, b)$ and hence $s+c \leq b+c \leq n$. Therefore, $\min(s+c-n+h, h) = s+c-n+h$. For simplicity, let us write $c' = c-n+h \geq 1$.

We claim that

$$(9.3) \quad \sum_{i=s}^{s+k} \frac{X_{h,h}(c', i, s)}{X_{h,h}(c', s, s)} = (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c'-k}^{c'-1} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l})},$$

which specializes to

$$\sum_{i=s}^{s+c'} \frac{X_{h,h}(c', i, s)}{X_{h,h}(c', s, s)} = 0$$

when $k = c'$.

Therefore it suffices to prove (9.3). We prove this by induction on k . Recall that by definition, we have

$$\frac{X_{h,h}(c', s+k, s)}{X_{h,h}(c', s, s)} = (-1)^k (-q)^{-\frac{k(k-1)}{2}} \frac{\prod_{l=c'-k+1}^{c'} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l})}.$$

When $k = 0$, both sides are 1. When $k = 1$,

$$\sum_{i=s}^{s+1} \frac{X_{h,h}(c', i, s)}{X_{h,h}(c', s, s)} = 1 - \frac{1 - (-q)^{-c'}}{1 - (-q)^{-1}} = -(-q)^{-1} \frac{1 - (-q)^{-(c'-1)}}{1 - (-q)^{-1}}.$$

Now, assume that (9.3) holds for k . Then, we have

$$\begin{aligned}
\sum_{i=s}^{s+k+1} \frac{X_{h,h}(c', i, s)}{X_{h,h}(c', s, s)} &= (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c'-k}^{c'-1} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l})} + \frac{X_{h,h}(c', s+k+1, s)}{X_{h,h}(c', s, s)} \\
&= (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c'-k}^{c'-1} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l})} \left\{ 1 - \frac{1 - (-q)^{-c'}}{1 - (-q)^{-k-1}} \right\} \\
&= (-1)^{k+1} (-q)^{-\frac{(k+1)(k+2)}{2}} \frac{\prod_{l=c'-k-1}^{c'-1} (1 - (-q)^{-l})}{\prod_{l=1}^{k+1} (1 - (-q)^{-l})}.
\end{aligned}$$

This shows that (9.3) holds and finishes the proof of (9.1), and hence the proof of the lemma. \square

Lemma 9.2. For $c \geq 1$ and $h > s$, the function $X_{n,h}(c, i, s)$ (defined in the proof of the Lemma 9.1) satisfies

$$(9.4) \quad \sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} = \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \sum_{i=s}^{\min(s+c-1,h)} \frac{X_{n-1,h}(c-1, i, s)}{X_{n-1,h}(c-1, s, s)}.$$

Proof. To prove (9.4), we define $Z_{n,h}(c, j, s)$ to be

$$Z_{n,h}(c, j, s) = (-1)^{j-s} (-q)^{-\frac{(j-s)(j-s+1)}{2}} \frac{\prod_{l=c-j+s}^{c-1} (1 - (-q)^{-l}) \prod_{l=h-j+1}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^{j-s} (1 - (-q)^{-l}) \prod_{l=n-j+1}^{n-s} (1 - (-q)^{-l})}.$$

Note that if $c \geq 1$ and $h > s$, we have that $\min(s+c, h) > s$.

Now, we claim the following statement: for $k \geq 1$, we have

$$(9.5) \quad \sum_{i=s}^{s+k} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} - \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \sum_{i=s}^{s+k-1} \frac{X_{n-1,h}(c-1, i, s)}{X_{n-1,h}(c-1, s, s)} = Z_{n,h}(c, s+k, s).$$

By definition, we have that

$$\frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} = (-1)^{i-s} (-q)^{-\frac{(i-s)(i-s-1)}{2}} \frac{\prod_{l=c-i+s+1}^c (1 - (-q)^{-l}) \prod_{l=h-i+1}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^{i-s} (1 - (-q)^{-l}) \prod_{l=n-i+1}^{n-s} (1 - (-q)^{-l})}.$$

In particular, when $k = 1$, we have that

$$\begin{aligned}
\sum_{i=s}^{s+1} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} &= 1 - \frac{(1 - (-q)^{-c})(1 - (-q)^{-(h-s)})}{(1 - (-q)^{-1})(1 - (-q)^{-(n-s)})} \\
&= \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} - \frac{(-q)^{-1} (1 - (-q)^{-(c-1)}) (1 - (-q)^{-(h-s)})}{(1 - (-q)^{-1})(1 - (-q)^{-(n-s)})} \\
&= \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \frac{X_{n-1,h}(c-1, s, s)}{X_{n-1,h}(c-1, s, s)} + Z_{n,h}(c, s+1, s).
\end{aligned}$$

Therefore, (9.5) holds for $k = 1$.

Now, assume that (9.5) is true for k , then it suffices to show that

$$(9.6) \quad \begin{aligned} & Z_{n,h}(c, s+k, s) + \frac{X_{n,h}(c, s+k+1, s)}{X_{n,h}(c, s, s)} \\ &= \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \frac{X_{n-1,h}(c-1, s+k, s)}{X_{n-1,h}(c-1, s, s)} + Z_{n,h}(c, s+k+1, s). \end{aligned}$$

Indeed,

$$\begin{aligned}
& Z_{n,h}(c, s+k, s) + \frac{X_{n,h}(c, s+k+1, s)}{X_{n,h}(c, s, s)} \\
&= (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c-k}^{c-1} (1 - (-q)^{-l}) \prod_{l=h-s-k+1}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=n-s-k+1}^{n-s} (1 - (-q)^{-l})} \left\{ 1 - \frac{(1 - (-q)^{-c})(1 - (-q)^{-(h-s-k)})}{(1 - (-q)^{-(k+1)})(1 - (-q)^{-(-n-s-k)})} \right\},
\end{aligned}$$

and

$$\begin{aligned}
1 - \frac{(1 - (-q)^{-c})(1 - (-q)^{-(h-s-k)})}{(1 - (-q)^{-(k+1)})(1 - (-q)^{-(-n-s-k)})} &= \frac{(-q)^k (-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s-k)}} \\
&\quad - \frac{(-q)^{-(k+1)} (1 - (-q)^{-(-c-k-1)}) (1 - (-q)^{-(h-s-k)})}{(1 - (-q)^{-(k+1)})(1 - (-q)^{-(n-s-k)})}.
\end{aligned}$$

Now, note that

$$\begin{aligned}
& (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c-k}^{c-1} (1 - (-q)^{-l}) \prod_{l=h-s-k+1}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=n-s-k+1}^{n-s} (1 - (-q)^{-l})} \times \frac{(-q)^k (-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s-k)}} \\
&= (-1)^k (-q)^{-\frac{k(k-1)}{2}} \frac{\prod_{l=c-k}^{c-1} (1 - (-q)^{-l}) \prod_{l=h-s-k+1}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=n-s-k}^{n-s-1} (1 - (-q)^{-l})} \times \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \\
&= \frac{X_{n-1,h}(c-1, s+k, s)}{X_{n-1,h}(c-1, s, s)} \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}},
\end{aligned}$$

and

$$\begin{aligned}
& (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c-k}^{c-1} (1 - (-q)^{-l}) \prod_{l=h-s-k+1}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=n-s-k+1}^{n-s} (1 - (-q)^{-l})} \\
&\quad \times \left\{ - \frac{(-q)^{-(k+1)} (1 - (-q)^{-(-c-k-1)}) (1 - (-q)^{-(h-s-k)})}{(1 - (-q)^{-(k+1)})(1 - (-q)^{-(n-s-k)})} \right\} \\
&= (-1)^{k+1} (-q)^{-\frac{(k+1)(k+2)}{2}} \frac{\prod_{l=c-k-1}^{c-1} (1 - (-q)^{-l}) \prod_{l=h-s-k}^{h-s} (1 - (-q)^{-l})}{\prod_{l=1}^{k+1} (1 - (-q)^{-l}) \prod_{l=n-s-k}^{n-s} (1 - (-q)^{-l})} \\
&= Z_{n,h}(c, s+k+1, s).
\end{aligned}$$

Therefore, combining these two, we have that (9.6) holds, and hence (9.5) holds.

Now, we are ready to prove (9.4). When $h \geq s+c$, we have that $\min(s+c, h) = s+c$ and $\min(s+c-1, h) = s+c-1$. Also, by (9.5), we have

$$\sum_{i=s}^{s+c} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} - \sum_{i=s}^{s+c-1} \frac{X_{n-1,h}(c-1, i, s)}{X_{n-1,h}(c-1, s, s)} \frac{(-q)^{-(h-s)} (1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} = Z_{n,h}(c, s+c, s).$$

Now, note that $Z_{n,h}(c, s+c, s) = 0$. This implies that (9.4) holds when $h \geq s+c$.

When $h \leq s + c - 1$, we have that $\min(s + c, h) = h$ and $\min(s + c - 1, h) = h$. Therefore, by (9.5), we have

$$\begin{aligned}
& \sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} - \frac{(-q)^{-(h-s)}(1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \sum_{i=s}^{\min(s+c-1,h)} \frac{X_{n-1,h}(c-1, i, s)}{X_{n-1,h}(c-1, s, s)} \\
&= \sum_{i=s}^h \frac{X_{n,h}(c, i, s)}{X_{n,h}(c, s, s)} - \frac{(-q)^{-(h-s)}(1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \sum_{i=s}^h \frac{X_{n-1,h}(c-1, i, s)}{X_{n-1,h}(c-1, s, s)} \\
&= Z_{n,h}(c, h, s) - \frac{(-q)^{-(h-s)}(1 - (-q)^{-(n-h)})}{1 - (-q)^{-(n-s)}} \frac{X_{n-1,h}(c-1, h, s)}{X_{n-1,h}(c-1, s, s)} \\
&= 0.
\end{aligned}$$

This finishes the proof of the lemma. \square

Lemma 9.3. *Let $0 \leq a, b, c, h \leq n$, $a + b + c = n$, and $0 \leq s \leq \min(h, b)$. If $c \leq n - h$ and $(a, b, c) \neq (0, h, n - h)$, we have that*

$$\begin{aligned}
& \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) - \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a-1, b+1, c, i, s) \\
&= -(-q)^{2n-h-1-b-2c} \sum_{i=s-1}^{\min(s+c-1,h-1)} M_{n-1,h-1}(a-1, b, c, i, s-1).
\end{aligned}$$

Proof. By Lemma 9.1 (1), we have that

$$\begin{aligned}
& \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) - \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a-1, b+1, c, i, s) \\
&= (-1)^{n+c+s+1} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch + \frac{s}{2} - \frac{s^2}{2}} \\
& \quad \times \left\{ \frac{\prod_{l=1}^b (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l})}{\prod_{l=1}^{b-s} (1 - (-q)^{-l}) \prod_{l=1}^s (1 - (-q)^{-l}) \prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})} \right. \\
& \quad \left. - \frac{\prod_{l=1}^{b+1} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l})}{\prod_{l=1}^{b+1-s} (1 - (-q)^{-l}) \prod_{l=1}^s (1 - (-q)^{-l}) \prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})} \right\} \\
&= (-1)^{n+c+s} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch + \frac{s}{2} - \frac{s^2}{2} - (b+1-s)} \\
& \quad \times \frac{\prod_{l=1}^b (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l})}{\prod_{l=1}^{b+1-s} (1 - (-q)^{-l}) \prod_{l=1}^{s-1} (1 - (-q)^{-l}) \prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& -(-q)^{2n-h-1-b-2c} \sum_{i=s-1}^{\min(s+c-1,h-1)} M_{n-1,h-1}(a-1, b, c, i, s-1) \\
&= (-1)^{n+c+s} (-q)^{\frac{c}{2} + \frac{c^2}{2} - (h-1)^2 - \frac{n-1}{2} - c(n-1) + (h-1)(n-1) + \frac{(n-1)^2}{2} - c(h-1) + \frac{s-1}{2} - \frac{(s-1)^2}{2} + 2n-h-1-b-2c} \\
& \quad \times \frac{\prod_{l=1}^b (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l})}{\prod_{l=1}^{b+1-s} (1 - (-q)^{-l}) \prod_{l=1}^{s-1} (1 - (-q)^{-l}) \prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})} \\
&= (-1)^{n+c+s} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch + \frac{3s}{2} - \frac{s^2}{2} - b-1} \\
& \quad \times \frac{\prod_{l=1}^b (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-s-1} (1 - (-q)^{-l})}{\prod_{l=1}^{b+1-s} (1 - (-q)^{-l}) \prod_{l=1}^{s-1} (1 - (-q)^{-l}) \prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})}.
\end{aligned}$$

Now, it is easy to see that the above two are the same. This finishes the proof of the lemma.

□

Theorem 9.4. *Let $0 \leq a, b, c, h \leq n$ and $a + b + c = n$. If $c \leq n - h$, $1 \leq a$, and $(a - 1, b + 1, c) \neq (0, h, n - h)$, we have that*

$$D_{n,h}(a, b, c) - D_{n,h}(a - 1, b + 1, c) = -(-q)^{2n-h-1-b-2c} D_{n-1,h-1}(a - 1, b, c).$$

Proof. First, by Lemma 9.1, we have that

$$\begin{aligned} \sum_{i=0}^{\min(0+c,h)} M_{n,h}(a, b, c, i, 0) &= (-1)^{n+c+1} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch} \\ &\quad \frac{\prod_{l=1}^{n-c} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-1} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^h (1 - (-q)^{-l})}. \end{aligned}$$

Since this does not depend on a, b , we have that

$$\sum_{i=0}^{\min(0+c,h)} M_{n,h}(a, b, c, i, 0) = \sum_{i=0}^{\min(0+c,h)} M_{n,h}(a - 1, b + 1, c, i, 0).$$

Since

$$D_{n,h}(a, b, c) = \sum_{s=0}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s),$$

we have that

$$\begin{aligned} &D_{n,h}(a, b, c) - D_{n,h}(a - 1, b + 1, c) \\ &= \sum_{s=0}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) - \sum_{s=0}^{\min(h,b+1)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a - 1, b + 1, c, i, s) \\ &= \sum_{s=1}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) - \sum_{s=1}^{\min(h,b+1)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a - 1, b + 1, c, i, s). \end{aligned}$$

Now, assume that $h \leq b$. Then $\min(h, b) = h$, $\min(h, b + 1) = h$, $\min(h - 1, b) = h - 1$. Therefore, by Lemma 9.3, we have

$$\begin{aligned} &D_{n,h}(a, b, c) - D_{n,h}(a - 1, b + 1, c) \\ &= \sum_{s=1}^h \left\{ \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) - \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a - 1, b + 1, c, i, s) \right\} \\ &= \sum_{s=0}^{h-1} -(-q)^{2n-h-1-b-2c} \sum_{i=s}^{\min(s+c,h-1)} M_{n-1,h-1}(a - 1, b, c, i, s) \\ &= -(-q)^{2n-h-1-b-2c} D_{n-1,h-1}(a - 1, b, c). \end{aligned}$$

This finishes the proof of the theorem when $h \leq b$.

Now, assume that $h > b$. Then $\min(h, b) = b$, $\min(h, b + 1) = b + 1$. Therefore, by Lemma 9.3, we have that

$$\begin{aligned}
& D_{n,h}(a, b, c) - D_{n,h}(a - 1, b + 1, c) \\
&= \sum_{s=1}^b \left\{ \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) - \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a - 1, b + 1, c, i, s) \right\} \\
&\quad - \sum_{i=b+1}^{\min(b+1+c,h)} M_{n,h}(a - 1, b + 1, c, i, b + 1) \\
(9.7) \quad &= \sum_{s=0}^{b-1} -(-q)^{2n-h-1-b-2c} \sum_{i=s}^{\min(s+c,h-1)} M_{n-1,h-1}(a - 1, b, c, i, s) \\
&\quad - \sum_{i=b+1}^{\min(b+1+c,h)} M_{n,h}(a - 1, b + 1, c, i, b + 1).
\end{aligned}$$

By Lemma 9.1, we have that

$$\begin{aligned}
& - \sum_{i=b+1}^{\min(b+1+c,h)} M_{n,h}(a - 1, b + 1, c, i, b + 1) \\
&= -(-1)^{n+c+b} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch + \frac{(b+1)}{2} - \frac{(b+1)^2}{2}} \\
&\quad \times \frac{\prod_{l=1}^{n-c-b-1} (1 - (-q)^{-l}) \prod_{l=1}^{n-c-b-2} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-b-1} (1 - (-q)^{-l})} \\
&= -(-1)^{n-1+c+b+1} (-q)^{2n-h-1-b-2c + \frac{c}{2} + \frac{c^2}{2} - (h-1)^2 - \frac{n-1}{2} - c(n-1) + (h-1)(n-1) + \frac{(n-1)^2}{2} - c(h-1) + \frac{b}{2} - \frac{b^2}{2}} \\
&\quad \times \frac{\prod_{l=1}^{n-1-c-b} (1 - (-q)^{-l}) \prod_{l=1}^{n-1-c-b-1} (1 - (-q)^{-l})}{\prod_{l=1}^{(n-1)-(h-1)-c} (1 - (-q)^{-l}) \prod_{l=1}^{h-1-b} (1 - (-q)^{-l})} \\
&= -(-q)^{2n-h-1-b-2c} \sum_{i=b}^{\min(b+c,h-1)} M_{n-1,h-1}(a - 1, b, c, i, b).
\end{aligned}$$

Therefore, since $\min(h - 1, b) = b$, (9.7) can be written as

$$\begin{aligned}
& D_{n,h}(a, b, c) - D_{n,h}(a - 1, b + 1, c) \\
&= -(-q)^{2n-h-1-b-2c} \sum_{s=0}^b \sum_{i=s}^{\min(s+c,h-1)} M_{n-1,h-1}(a - 1, b, c, i, s) \\
&= -(-q)^{2n-h-1-b-2c} D_{n-1,h-1}(a - 1, b, c).
\end{aligned}$$

This finishes the proof of the theorem. \square

Lemma 9.5. *Let $0 \leq a, b, c, h \leq n$, and $a + b + c = n$. If $c > n - h$ and $a \neq 0$, we have that*

$$D_{n,h}(a, b, c) = 0.$$

Therefore, for $\lambda \in \mathcal{R}_n^{0+}$ such that $t_0(\lambda) > n - h$, and $t_{\geq 2}(\lambda) \neq 0$, we have that

$$D_{n,h}(\lambda) = 0.$$

Proof. This follows from Lemma 9.1 (2), Definition 8.19 (3) and Proposition 8.20. \square

Proposition 9.6. *For $0 \leq t \leq h - 1$, $t \equiv h + 1 \pmod{2}$, and $\lambda = (1^t, 0^{n-t}) \in \mathcal{R}_n^{0+}$, we have that*

$$D_{n,h}(\lambda) = 0.$$

Proof. First, by Proposition 8.20, we have that

$$D_{n,h}(\lambda) = D_{n,h}(0, t, n-t) = \sum_{\substack{0 \leq s \leq \min(h,b) \\ s \leq i \leq \min(s+c,h)}} M_{n,h}(a, b, c, i, s) + \frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}}.$$

Before we start the proof of the proposition, let us say why we consider this case separately. Recall from Definition 8.19 (1) that $\mathcal{C}_l(0, 0, k)$ is defined separately. If λ is not of the form $(1^t, 0^{n-t})$, $t < h$, these constants $\mathcal{C}_l(0, 0, k)$'s do not appear in the sum $\sum_{\substack{0 \leq s \leq \min(h,b) \\ s \leq i \leq \min(s+c,h)}} M_{n,h}(a, b, c, i, s)$. However,

if $\lambda = (1^t, 0^{n-t})$, $t < h$, $M_{n,h}(0, t, n-t, i, t)$ has the term $\mathcal{C}_{h+1-t}(0, 0, n-i)$. Therefore, we need to be careful when we compute $D_{n,h}(0, t, n-t)$.

Now, let us start the proof of the proposition. Recall from the proof of the Lemma 9.1 that we have

$$M_{n,h}(a, b, c, i, s) = \mathcal{C}_{h+1-s}(a, b-s, c+s-i) Y_{n,h}(b, c, s) X_{n,h}(c, i, s),$$

Also, by (9.1), we have that

$$\sum_{i=s}^{\min(s+n-t,h)} Y_{n,h}(t, n-t, s) X_{n,h}(n-t, i, s) = Y_{n,h}(t, n-t, s) \sum_{i=s}^{\min(s+n-t,h)} X_{n,h}(n-t, i, s) = 0$$

Also, note that $\min(h, t) = t$ and $\mathcal{C}_{h+1-s}(0, t-s, n-t+s-i)$ does not depend on i if $s \neq t$. Therefore, we have that

$$\begin{aligned} & D_{n,h}(0, t, n-t) - \frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}} \\ &= \sum_{\substack{0 \leq s \leq \min(h,t) \\ s \leq i \leq \min(s+n-t,h)}} \mathcal{C}_{h+1-s}(0, t-s, n-t+s-i) Y_{n,h}(t, n-t, s) X_{n,h}(n-t, i, s) \\ &= \sum_{s=0}^{t-1} \mathcal{C}_{h+1-s}(0, t-s, n-t+s) Y_{n,h}(t, n-t, s) \sum_{i=s}^{\min(s+n-t,h)} X_{n,h}(n-t, i, s) \\ &\quad + \sum_{i=t}^h \mathcal{C}_{h+1-t}(0, 0, n-i) Y_{n,h}(t, n-t, t) X_{n,h}(n-t, i, t) \\ &= \sum_{i=t}^h \mathcal{C}_{h+1-t}(0, 0, n-i) Y_{n,h}(t, n-t, t) X_{n,h}(n-t, i, t). \end{aligned}$$

Now, recall that $\mathcal{C}_{h+1-t}(0, 0, n-i) = \frac{\text{Den}'(I_{n-i}, I_{n-i})}{\text{Den}(I_{n-i}, I_{n-i})} = \sum_{l=1}^{n-i} \frac{1}{(-q)^l - 1}$.

This implies that it suffices to show that

$$(9.8) \quad Y_{n,h}(t, n-t, t) \sum_{i=t}^h \left(\sum_{l=1}^{n-i} \frac{1}{(-q)^l - 1} \right) X_{n,h}(n-t, i, t) = -\frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}}.$$

First, we have that

$$\begin{aligned} \sum_{i=t}^h \left(\sum_{l=1}^{n-i} \frac{1}{(-q)^l - 1} \right) X_{n,h}(n-t, i, t) &= \sum_{i=t}^h X_{n,h}(n-t, i, t) \left(\sum_{l=1}^{n-h} \frac{1}{(-q)^l - 1} \right) \\ &+ \sum_{k=0}^{h-t-1} \frac{1}{(-q)^{n-t-k} - 1} \sum_{i=t}^{t+k} X_{n,h}(n-t, i, t). \end{aligned}$$

Since $\sum_{i=t}^h X_{n,h}(n-t, i, t) = 0$ (by (9.1)), we have that

$$(9.9) \quad \sum_{i=t}^h \left(\sum_{l=1}^{n-i} \frac{1}{(-q)^l - 1} \right) X_{n,h}(n-t, i, t) = \sum_{k=0}^{h-t-1} \frac{1}{(-q)^{n-t-k} - 1} \sum_{i=t}^{t+k} X_{n,h}(n-t, i, t).$$

We claim that

$$(9.10) \quad \sum_{i=t}^{t+k} X_{n,h}(n-t, i, t) = \frac{(-1)^{t+k} (-q)^{\frac{(t+k+1)(t-k)}{2}}}{\prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l}) (1 - (-q)^{-(h-t)})}.$$

Indeed, it is easy to see that this holds for $k = 0$. Now, assume that this holds for k . Then, we have that

$$\begin{aligned} \sum_{i=t}^{t+k+1} X_{n,h}(n-t, i, t) &= \sum_{i=t}^{t+k+1} \frac{(-1)^i (-q)^{\frac{i}{2} - \frac{i^2}{2} + it}}{\prod_{l=1}^{h-i} (1 - (-q)^{-l}) \prod_{l=1}^{i-t} (1 - (-q)^{-l})} \\ &= \frac{(-1)^{t+k} (-q)^{\frac{(t+k+1)(t-k)}{2}}}{\prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l}) (1 - (-q)^{-(h-t)})} + \frac{(-1)^{t+k+1} (-q)^{\frac{t+k+1}{2} - \frac{(t+k+1)^2}{2} + (t+k+1)t}}{\prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^{k+1} (1 - (-q)^{-l})} \\ &= \frac{(-1)^{t+k+1} (-q)^{\frac{(t+k+1)(t-k)}{2}}}{\prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^{k+1} (1 - (-q)^{-l})} \left\{ -\frac{(1 - (-q)^{-(k+1)})}{1 - (-q)^{-(h-t)}} + 1 \right\} \\ &= \frac{(-1)^{t+k+1} (-q)^{\frac{(t+k+1)(t-k)}{2}}}{\prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^{k+1} (1 - (-q)^{-l})} \frac{(-q)^{-(k+1)} (1 - (-q)^{-h-t+k+1})}{(1 - (-q)^{-(h-t)})} \\ &= \frac{(-1)^{t+k+1} (-q)^{\frac{(t+k+2)(t-k-1)}{2}}}{\prod_{l=1}^{h-t-k-2} (1 - (-q)^{-l}) \prod_{l=1}^{k+1} (1 - (-q)^{-l}) (1 - (-q)^{-(h-t)})}. \end{aligned}$$

This shows that (9.10) holds.

Now, by (9.10), we have that (9.9) can be written as

$$(9.11) \quad \begin{aligned} \sum_{i=t}^h \left(\sum_{l=1}^{n-i} \frac{1}{(-q)^l - 1} \right) X_{n,h}(n-t, i, t) &= \sum_{k=0}^{h-t-1} \frac{1}{(-q)^{n-t-k} - 1} \sum_{i=t}^{t+k} X_{n,h}(n-t, i, t) \\ &= \frac{(-1)^t (-q)^{\frac{t^2}{2} + \frac{3t}{2} - n}}{(1 - (-q)^{-(h-t)})} \times \sum_{k=0}^{h-t-1} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}}}{(1 - (-q)^{-(n-t-k)}) \prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})}. \end{aligned}$$

In Lemma 9.7 below, we will prove that

$$(9.12) \quad \begin{aligned} & \sum_{k=0}^{h-t-1} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}}}{(1 - (-q)^{-(n-t-k)}) \prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})} \\ &= \frac{(-1)^{h-t-1} (-q)^{-\frac{(h-t)(h-t-1)}{2}} (-q)^{-(n-h)(h-t-1)}}{\prod_{l=n-h+1}^{n-t} (1 - (-q)^{-l})}. \end{aligned}$$

Combining (9.11) and (9.12), we have that

$$\begin{aligned} & Y_{n,h}(t, n-t, t) \sum_{i=t}^h \left(\sum_{l=1}^{n-i} \frac{1}{(-q)^l - 1} \right) X_{n,h}(n-t, i, t) \\ &= \frac{(-1)^h (-q)^{-h^2 + hn - \frac{t}{2} + \frac{t^2}{2} - nt} \prod_{l=1}^{n-t} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h} (1 - (-q)^{-l})} \\ & \times \frac{(-1)^t (-q)^{\frac{t^2}{2} + \frac{3t}{2} - n}}{(1 - (-q)^{-(h-t)})} \times \frac{(-1)^{h-t-1} (-q)^{-\frac{(h-t)(h-t-1)}{2}} (-q)^{-(n-h)(h-t-1)}}{\prod_{l=n-h+1}^{n-t} (1 - (-q)^{-l})} \\ &= \frac{-(-q)^{-\frac{(h-t)(h+t+1)}{2}}}{1 - (-q)^{-(h-t)}}. \end{aligned}$$

This finishes the proof of the proposition. □

Lemma 9.7. For $0 \leq t < h \leq n$, we have

$$\begin{aligned} & \sum_{k=0}^{h-t-1} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}}}{(1 - (-q)^{-(n-t-k)}) \prod_{l=1}^{h-t-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})} \\ &= \frac{(-1)^{h-t-1} (-q)^{-\frac{(h-t)(h-t-1)}{2}} (-q)^{-(n-h)(h-t-1)}}{\prod_{l=n-h+1}^{n-t} (1 - (-q)^{-l})}. \end{aligned}$$

Proof. For $N > 0$, we claim the following statement:

$$(9.13) \quad \sum_{k=0}^{N-1} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}}}{(1 - (-q)^{-(N-k)} X) \prod_{l=1}^{N-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})} = \frac{(-1)^{N-1} (-q)^{-\frac{N(N-1)}{2}} X^{N-1}}{\prod_{l=1}^N (1 - (-q)^{-l} X)},$$

which specializes to the statement of the lemma when $X = (-q)^{-(n-h)}$. Therefore it suffices to show the claim (9.13).

Consider a Vandermonde matrix

$$\mathfrak{X} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{pmatrix},$$

and let $\mathfrak{X}^{-1} = (y_{ij})_{1 \leq i, j \leq N}$. Note that y_{ij} is the X^{N-j} -coefficient of

$$\prod_{l=1, l \neq i}^N \frac{1 - x_l X}{x_i - x_l}.$$

Now, assume that $x_l = (-q)^{-l}$. Then, we have that $y_{(N-k)j}$ is the X^{N-j} -coefficient of

$$\begin{aligned} \prod_{l=1, l \neq (N-k)}^N \frac{(1 - (-q)^{-l} X)}{(-q)^{-(N-k)} - (-q)^{-l}} &= \frac{(-1)^{N-k-1} (-q)^{\frac{(N-k)(N+k-1)}{2}} \prod_{l=1, l \neq (N-k)}^N (1 - (-q)^{-l} X)}{\prod_{l=1}^{N-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})} \\ &= (-1)^{N-1} (-q)^{\frac{N(N-1)}{2}} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}} \prod_{l=1, l \neq (N-k)}^N (1 - (-q)^{-l} X)}{\prod_{l=1}^{N-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})}. \end{aligned}$$

Therefore, we have that

$$(9.14) \quad \sum_{j=1}^N \sum_{k=0}^{N-1} y_{(N-k)j} X^{N-j} = (-1)^{N-1} (-q)^{\frac{N(N-1)}{2}} \sum_{k=0}^{N-1} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}} \prod_{l=1, l \neq N-k}^N (1 - (-q)^{-l} X)}{\prod_{l=1}^{N-k-1} (1 - (-q)^{-l}) \prod_{l=1}^k (1 - (-q)^{-l})}.$$

Now, note that

$$(1 \ 0 \dots 0) \mathfrak{X} = (1 \ 1 \dots 1) \iff (1 \ 0 \dots 0) = (1 \ 1 \dots 1) \mathfrak{X}^{-1} = \left(\sum_{i=1}^N y_{i1} \quad \sum_{i=1}^N y_{i2} \dots \sum_{i=1}^N y_{iN} \right).$$

Hence, we have

$$(9.15) \quad \sum_{j=1}^N \sum_{k=0}^{N-1} y_{(N-k)j} X^{N-j} = X^{N-1}.$$

Therefore, (9.14) and (9.15) imply that (9.13) holds. This finishes the proof of the lemma. \square

Now, let us state the following theorem.

Theorem 9.8. *Assume that $0 \leq h \leq n$.*

(1) *Assume that $h + 1 \leq b \leq n$, and $b + c = n$. Then, we have*

$$D_{n,h}(0, b, c) = \frac{\prod_{l=h+1}^b (1 - (-q)^l)}{(1 - (-q)^{b-h})}.$$

(2) *Assume that $h - 1 \leq b \leq n$, and $b + c = n - 1$. Then, we have*

$$D_{n,h}(1, b, c) = \begin{cases} 1 & \text{if } b = h - 1, h, \\ \prod_{l=h+1}^b (1 - (-q)^l) & \text{if } b \geq h + 1. \end{cases}$$

Proof. Recall from Lemma 9.1 that for $a = 0$ or 1 (i.e., $b + c = n$ or $n - 1$, respectively), we have

$$(9.16) \quad \begin{aligned} D_{n,h}(a, b, c) &= \sum_{s=0}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a, b, c, i, s) \\ &= (-1)^{n+c+1} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch} \frac{\prod_{l=1}^{n-c-a} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h-c} (1 - (-q)^{-l})} \\ &\quad \times \sum_{s=0}^{\min(h,b)} \frac{(-1)^s (-q)^{\frac{s}{2} - \frac{s^2}{2}} \prod_{l=1}^{n-c-s+a-1} (1 - (-q)^{-l})}{\prod_{l=1}^s (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})}. \end{aligned}$$

First, assume that $a = 1$, $b = h - 1$, and $c = n - h$. Then, we have

$$\sum_{s=0}^{\min(h,b)} \frac{(-1)^s (-q)^{\frac{s}{2} - \frac{s^2}{2}} \prod_{l=1}^{n-c-s+a-1} (1 - (-q)^{-l})}{\prod_{l=1}^s (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})} = \sum_{s=0}^{h-1} \frac{(-1)^s (-q)^{\frac{s}{2} - \frac{s^2}{2}}}{\prod_{l=1}^s (1 - (-q)^{-l})} = \frac{(-1)^{h-1} (-q)^{-\frac{h(h-1)}{2}}}{\prod_{l=1}^{h-1} (1 - (-q)^{-l})}.$$

Therefore, we have

$$\begin{aligned} D_{n,h}(1, h-1, n-h) &= (-1)^{n+c+h} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch - h(h-1)/2} \frac{\prod_{l=1}^{h-1} (1 - (-q)^{-l})}{\prod_{l=1}^{h-1} (1 - (-q)^{-l})} \\ &= (-q)^{\frac{(n-c-h)(n-c+3h-1)}{2}} = 1. \end{aligned}$$

Now, assume that $b \geq h$. In this case, we have that $\min(h, b) = h$. We claim that

$$(9.17) \quad \sum_{s=0}^h \frac{(-1)^s (-q)^{\frac{s}{2} - \frac{s^2}{2}} \prod_{l=1}^{N-s} (1 - (-q)^{-l})}{\prod_{l=1}^s (1 - (-q)^{-l}) \prod_{l=1}^{h-s} (1 - (-q)^{-l})} = \frac{(-1)^h (-q)^{-hN + \frac{h(h-1)}{2}} \prod_{l=1}^{N-h} (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l})},$$

where $N = n - c + a - 1$.

To prove (9.17), we define the following constants: for $0 \leq k \leq h$, $1 \leq t \leq k+1$,

$$\omega_{k,t} = \begin{cases} \frac{(-1)^k (-q)^{-(t-1)N + \frac{(t-1)(t-2)}{2} - \frac{(k-t+1)(k-t+2)}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^{k-t+1} (1 - (-q)^{-l}) \prod_{l=1}^{h-k-1} (1 - (-q)^{-l}) \prod_{l=h-t+1}^h (1 - (-q)^{-l})} & \text{if } k \leq h-1, \\ 0 & \text{if } k = h, t \neq h+1, \\ \frac{(-1)^h (-q)^{-hN + \frac{h(h-1)}{2}} \prod_{l=1}^{N-h} (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l})} & \text{if } k = h, t = h+1, \end{cases}$$

and

$$\tau_{k,t} = \frac{(-1)^k (-q)^{-tN + \frac{t(t-1)}{2} - \frac{(k-t-1)(k-t)}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^{k-t} (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l}) \prod_{l=h-t+1}^h (1 - (-q)^{-l})}.$$

Note that

$$\omega_{0,1} = \frac{\prod_{l=1}^N (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l})} = \frac{(-1)^0 (-q)^{\frac{0}{2} - \frac{0^2}{2}} \prod_{l=1}^{N-0} (1 - (-q)^{-l})}{\prod_{l=1}^0 (1 - (-q)^{-l}) \prod_{l=1}^{h-0} (1 - (-q)^{-l})},$$

and

$$\tau_{k,k} = \frac{(-1)^k (-q)^{-kN + \frac{k(k-1)}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l})} = \omega_{k,k+1}.$$

We claim the following two equations: for $k \geq 1$, $2 \leq t \leq k$, we have

$$(9.18) \quad \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} + \omega_{k-1,1} = \omega_{k,1} + \tau_{k,1},$$

and

$$(9.19) \quad \tau_{k,t-1} + \omega_{k-1,t} = \omega_{k,t} + \tau_{k,t}.$$

First, we have

$$\begin{aligned} & \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} + \omega_{k-1,1} \\ &= \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^{k-1} (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} \left\{ \frac{1}{1 - (-q)^{-k}} - \frac{1 - (-q)^{-(N-k+1)}}{1 - (-q)^{-h}} \right\} \\ &= \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^{k-1} (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} \left\{ \frac{(-q)^{-k} (1 - (-q)^{-(h-k)})}{(1 - (-q)^{-k})(1 - (-q)^{-h})} + \frac{(-q)^{-(N-k+1)}}{1 - (-q)^{-h}} \right\} \\ &= \omega_{k,1} + \tau_{k,1}. \end{aligned}$$

This shows that (9.18) holds.

For (9.19), we have

$$\begin{aligned}
\tau_{k,t-1} + \omega_{k-1,t} &= \frac{(-1)^k (-q)^{-(t-1)N + \frac{(t-1)(t-2)}{2} - \frac{(k-t)(k-t+1)}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^{k-t} (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l}) \prod_{l=h-t+2}^h (1 - (-q)^{-l})} \\
&\quad \times \left\{ \frac{1}{(1 - (-q)^{-(k-t+1)})} - \frac{1 - (-q)^{-(N-k+1)}}{1 - (-q)^{-(h-t+1)}} \right\} \\
&= \frac{(-1)^k (-q)^{-(t-1)N + \frac{(t-1)(t-2)}{2} - \frac{(k-t)(k-t+1)}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^{k-t} (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l}) \prod_{l=h-t+2}^h (1 - (-q)^{-l})} \\
&\quad \times \left\{ \frac{(-q)^{-(k-t+1)} (1 - (-q)^{-(h-k)})}{(1 - (-q)^{-(k-t+1)}) (1 - (-q)^{-(h-t+1)})} + \frac{(-q)^{-(N-k+1)}}{1 - (-q)^{-(h-t+1)}} \right\} \\
&= \omega_{k,t} + \tau_{k,t}.
\end{aligned}$$

This shows that (9.19) holds. Now, by (9.18) and (9.19), we have that for $k \geq 1$,

$$\frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} + \sum_{t=1}^k \omega_{k-1,t} + \sum_{t=1}^{k-1} \tau_{k,t} = \sum_{t=1}^k \omega_{k,t} + \sum_{t=1}^k \tau_{k,t},$$

and hence

$$(9.20) \quad \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} + \sum_{t=1}^k \omega_{k-1,t} = \sum_{t=1}^k \omega_{k,t} + \tau_{k,k} = \sum_{t=1}^{k+1} \omega_{k,t}.$$

Here, we used $\tau_{k,k} = \omega_{k,k+1}$ for the last identity.

Therefore, by (9.20), we have

$$\begin{aligned}
&\sum_{k=1}^h \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} + \sum_{k=1}^h \sum_{t=1}^k \omega_{k-1,t} = \sum_{k=1}^h \sum_{t=1}^{k+1} \omega_{k,t} \\
\iff &\sum_{k=1}^h \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} + \omega_{0,1} = \sum_{t=1}^{h+1} \omega_{h,t} \\
\iff &\sum_{k=0}^h \frac{(-1)^k (-q)^{\frac{k}{2} - \frac{k^2}{2}} \prod_{l=1}^{N-k} (1 - (-q)^{-l})}{\prod_{l=1}^k (1 - (-q)^{-l}) \prod_{l=1}^{h-k} (1 - (-q)^{-l})} = \omega_{h,h+1} \quad (\text{since } \omega_{h,t} = 0 \text{ for all } t < h+1) \\
&= \frac{(-1)^h (-q)^{-hN + \frac{h(h-1)}{2}} \prod_{l=1}^{N-h} (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l})}.
\end{aligned}$$

This shows that (9.17) holds.

Combining (9.16) and (9.17), we have

$$\begin{aligned}
D_{n,h}(a, b, c) &= (-1)^{n+c+h+1} (-q)^{\frac{c}{2} + \frac{c^2}{2} - h^2 - \frac{n}{2} - cn + hn + \frac{n^2}{2} - ch - h(n-c+a-1) + \frac{h(h-1)}{2}} \\
&\quad \times \frac{\prod_{l=1}^{n-c-a} (1 - (-q)^{-l}) \prod_{l=1}^{n-c+a-1-h} (1 - (-q)^{-l})}{\prod_{l=1}^{n-h-c} (1 - (-q)^{-l}) \prod_{l=1}^h (1 - (-q)^{-l})}.
\end{aligned}$$

If $a = 1$, then we have $b = n - c - 1$, and

$$\begin{aligned} D_{n,h}(1, b, c) &= (-1)^{n+c+h+1} (-q)^{\frac{(n-c-h-1)(n-c+h)}{2}} \frac{\prod_{l=1}^b (1 - (-q)^{-l})}{\prod_{l=1}^h (1 - (-q)^{-l})} \\ &= \prod_{l=h+1}^b (1 - (-q)^l). \end{aligned}$$

If $a = 0$, then we have $b = n - c$, and

$$\begin{aligned} D_{n,h}(0, b, c) &= (-1)^{n+c+h+1} (-q)^{\frac{(n-c-h)(n-c+h-1)}{2}} \frac{\prod_{l=1}^b (1 - (-q)^{-l})}{(1 - (-q)^{-(n-c-h)}) \prod_{l=1}^h (1 - (-q)^{-l})} \\ &= \frac{\prod_{l=h+1}^b (1 - (-q)^l)}{(1 - (-q)^{b-h})}. \end{aligned}$$

This finishes the proof of the theorem. \square

10. FOURIER TRANSFORM

In this section, we will prove certain theorems on the Fourier transform of the analytic side of Conjecture 7.6. The main results of this section are Theorem 10.18, Theorem 10.19, Theorem 10.20, and Theorem 10.21. Recall that \mathbb{V} is the space of special homomorphisms associated with $\mathcal{N}_n^{[h]}$ and it is split (resp. non-split) if h is odd (resp. even). For an integrable function f on \mathbb{V} , we denote by \widehat{f} its Fourier transform

$$\widehat{f}(x) := \int_{\mathbb{V}} f(y) \psi(\mathrm{Tr}_{F/F_0} \langle x, y \rangle) dy, \quad x \in \mathbb{V}.$$

For example, for an O_F -lattice $L \in \mathbb{V}$ of rank n , we have

$$\widehat{1}_L = \mathrm{vol}(L) 1_{L^\vee} = q^{-\mathrm{val}(L)/2} 1_{L^\vee},$$

where we recall that $\mathrm{val}(L)$ is the sum of fundamental invariants of L .

Definition 10.1. Let $L \subset \mathbb{V}$ be an O_F -lattice of rank n and let $L^b \subset \mathbb{V}$ be an O_F -lattice of rank $n - 1$.

(1) For simplicity, we set

$$D_{n,h}(L) := \partial \mathrm{Pden}_{n,h}(L).$$

Assume that $\lambda \in \mathcal{R}_n^{0+}$ be the fundamental invariants of L . Then by Corollary 8.7, we have

$$D_{n,h}(L) = D_{n,h}(\lambda),$$

where $D_{n,h}(\lambda)$ is the Cho-Yamauchi constant defined in Definition 8.3.

(2) For $x \in \mathbb{V} \setminus L_F^b$, we define

$$\partial \mathrm{Den}_{L^b}^{n,h}(x) := \sum_{L^b \subset L' \subset L'^\vee} D_{n,h}(L') 1_{L'}(x),$$

where $L' \subset \mathbb{V}$ are O_F -lattices of rank n .

(3) For $x \in \mathbb{V} \setminus L_F^b$, and $L^b \supset L^b$, we define

$$\partial \widehat{\text{Den}}_{L^{b \circ}}^{n,h}(x) := \sum_{L^b \subset L' \subset L^\vee, L' \cap L_F^b = L^b} D_{n,h}(L') 1_{L'}(x),$$

where $L' \subset \mathbb{V}$ are O_F -lattices of rank n .

Then, we have that

$$\partial \widehat{\text{Den}}_{L^b}^{n,h}(x) = \sum_{L^b \subset L' \subset (L^b)^\vee} \partial \widehat{\text{Den}}_{L^{b \circ}}^{n,h}(x).$$

By definition, we have that

$$\begin{aligned} \widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) &= \sum_{L^b \subset L' \subset L^\vee, L' \cap L_F^b = L^b} D_{n,h}(L') \widehat{1}_{L'}(x) \\ &= \sum_{L^b \subset L' \subset L^\vee, L' \cap L_F^b = L^b, x \in L'^\vee} D_{n,h}(L') \text{vol}(L'). \end{aligned}$$

Also, by [LZ22a, Lemma 7.2.1, Lemma 7.2.2, (7.4.2.3)], we have that

$$\begin{array}{ccc} [(\langle x \rangle + L^b)^{\vee, \geq 0} / L^b] \setminus (L_F^b / L^b) & \xrightarrow{\sim} & \{L^b \subset L' \subset L^\vee, L' \cap L_F^b = L^b, x \in L'^\vee\} \\ u & \mapsto & L^b + \langle u \rangle. \end{array}$$

Now, assume that $x \perp L^b$ and $\text{val}(\langle x, x \rangle) \leq -1$. Then, as in the proof of [LZ22a, Lemma 7.4.2], we have that

$$(\langle x \rangle + L^b)^{\vee, \geq 0} = (L^b)^{\vee, \geq 0} \oplus \langle x \rangle^\vee,$$

and

$$[(\langle x \rangle + L^b)^{\vee, \geq 0} / L^b] \setminus (L_F^b / L^b) \xrightarrow{\sim} (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times.$$

Therefore, we have that

$$\begin{array}{ccc} (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times & \xrightarrow{\sim} & \{L^b \subset L' \subset L^\vee, L' \cap L_F^b = L^b, x \in L'^\vee\} \\ (u^b, u^\perp) & \mapsto & L^b + \langle u^b + u^\perp \rangle, \end{array}$$

and

$$(10.1) \quad \widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) = \sum_{(u^b, u^\perp) \in (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle).$$

Now, we need to compute $D_{n,h}(L^b + \langle u^b + u^\perp \rangle)$. First, let us define the following notations.

Definition 10.2. For $\lambda \in \mathcal{R}_{n-1}^{0+}$, we let $L_\lambda \subset \mathbb{V}$ be an O_F -lattice of rank $n-1$ with hermitian matrix A_λ . Consider a basis $\mathbb{B} = \{x_1, \dots, x_{n-1}\}$ such that the hermitian matrix of L_λ with respect to \mathbb{B} is A_λ .

(1) For $i \geq 0$, we define $L_{\lambda \geq i}$ to be the sublattice of L_λ generated by $\{x_1, \dots, x_{t_{\geq i}(\lambda)}\}$. Therefore, the hermitian matrix of $L_{\lambda \geq i}$ is

$$\begin{pmatrix} \pi^{\lambda_1} & & & & \\ & \pi^{\lambda_2} & & & \\ & & \ddots & & \\ & & & \pi^{\lambda_{t_{\geq i}(\lambda)}} & \\ & & & & \end{pmatrix}.$$

(7) (Case 3-2) If $u^b \in ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus (\pi(L_1^b)^\vee))^{\geq 0} - ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus (\pi(L_1^b)^\vee))^{\geq 1}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b, c + 1).$$

(8) (Case 4-1) If $u^b \in (\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a + 1, b - 2, c + 2).$$

(9) (Case 4-2) If

$u^b \in ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee))^{\geq 0} - (\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b - 1, c + 2).$$

(10) (Case 5) If $u^b \in (((L_2^b)^\vee - \pi(L_2^b)^\vee) \oplus (L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a - 1, b, c + 2).$$

Proof. This is a linear algebra problem, so we skip the proof of the proposition. \square

Proposition 10.4. Assume that $\text{val}((u^\perp, u^\perp)) = 1$, then we have the followings.

(1) (Case 1-1) If $u^b \in (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 2}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b + 1, c).$$

(2) (Case 1-2-1) If $u^b \in (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 1} - (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 2}$, and $\text{val}(\langle u^b + u^\perp, u^b + u^\perp \rangle) = 1$

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b + 1, c).$$

(3) (Case 1-2-2) If $u^b \in (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 1} - (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 2}$, and $\text{val}(\langle u^b + u^\perp, u^b + u^\perp \rangle) \geq 2$, we have

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a + 1, b, c).$$

(4) (Case 1-3) If $u^b \in (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 0} - (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 1}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b, c + 1).$$

(5) (Case 2-1) If $u^b \in (\pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a + 1, b - 2, c + 2).$$

(6) (Case 2-2) If

$u^b \in ((\pi^2(L_2^b)^\vee) \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee))^{\geq 0} - (\pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b - 1, c + 2).$$

(7) (Case 3-1) If $u^b \in ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus (\pi(L_1^b)^\vee))^{\geq 1}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a - 1, b + 2, c).$$

(8) (Case 3-2) If $u^b \in ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus (\pi(L_1^b)^\vee))^{\geq 0} - ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus (\pi(L_1^b)^\vee))^{\geq 1}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b, c + 1).$$

(9) (Case 4-1) If $u^b \in (\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a + 1, b - 2, c + 2).$$

(10) (Case 4-2) If

$$u^b \in ((\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee) \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee))^{\geq 0} - (\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0},$$

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a, b-1, c+2).$$

(11) (Case 5) If $u^b \in ((L_2^b)^\vee - \pi(L_2^b)^\vee) \oplus (L_1^b)^\vee)^{\geq 0}$,

$$D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = D_{n,h}(a-1, b, c+2).$$

Proof. This is a linear algebra problem, so we skip the proof of the proposition. \square

Remark 10.5. We note that the only difference between the case $\text{val}((u^\perp, u^\perp)) \geq 2$ (Proposition 10.3) and the case $\text{val}((u^\perp, u^\perp)) = 1$ (Proposition 10.4) is the cases 1-1, 1-2, and 1-3.

Proposition 10.6. Assume that $\lambda \in \mathcal{R}_n^{0+}$ and L_λ is an O_F -lattice with hermitian matrix A_λ .

(1) If $\lambda \geq (1, \dots, 1)$, we have

$$\mu^+(L_\lambda) = \mu(L_{\lambda-1}).$$

(2) If $\lambda \geq (2, \dots, 2)$, we have

$$\mu^{++}(L_\lambda) = \mu^+(L_{\lambda-1}).$$

Proof. By definition, we have that

$$\mu(L_{\lambda-1}) = |(L_{\lambda-1}^\vee)^{\geq 0}/L_{\lambda-1}|,$$

$$\mu^+(L_\lambda) = |(\pi L_\lambda^\vee)^{\geq 1}/L_\lambda|.$$

Note that the fundamental invariants of $L_{\lambda-1}^\vee$ (resp. πL_λ^\vee) are $(-\lambda_1 + 1, \dots, -\lambda_n + 1)$ (resp. $(-\lambda_1 + 2, \dots, -\lambda_n + 2)$). Now, for $(L_{\lambda-1}^\vee)^{\geq 0}/L_{\lambda-1}$, and the hermitian form $\langle \cdot, \cdot \rangle$, we consider the same set with the hermitian form $\pi \langle \cdot, \cdot \rangle$. Then, this is isomorphic to $(\pi L_\lambda^\vee)^{\geq 1}/L_\lambda$. This proves (1). The proof of (2) is similar. \square

Proposition 10.7. For $\lambda \in \mathcal{R}_{n-1}^{0+}$ and $L_\lambda = L_{\lambda \geq 2} \oplus L_{\lambda=1} =: L_2 \oplus L_1$, we have the followings.

(1) (Case 1-1) We have that

$$|(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 2}/L_2 \oplus L_1| = \mu(L_{(\lambda \geq 2)-2}).$$

(2) (Case 1-2) We have that

$$|\{(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 1} - (\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 2}\}/L_2 \oplus L_1|$$

$$= q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - \mu(L_{(\lambda \geq 2)-2}).$$

(3) (Case 1-3+Case 3-2) We have that

$$|\{(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 0} - (\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 1}\}/L_2 \oplus L_1|$$

$$+ |\{((\pi(L_2)^\vee - \pi^2(L_{\lambda \geq 2})^\vee) \oplus (\pi(L_1)^\vee))^{\geq 0} - ((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus (\pi(L_1)^\vee))^{\geq 1}\}/L_2 \oplus L_1|$$

$$= q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) - \mu(L_{(\lambda \geq 2)-1}).$$

(4) (Case 2-1+Case 4-1) We have that

$$|\{(\pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}\}/L_2 \oplus L_1|$$

$$+ |\{(\pi(L_2^b)^\vee - \pi^2(L_2^b)^\vee)^{\geq 0} \oplus ((L_1^b)^\vee - \pi(L_1^b)^\vee)^{\geq 0}\}/L_2 \oplus L_1|$$

$$= q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) \times (\mu(L_{\lambda=1}) - 1).$$

(5) (Case 2-2+Case 4-2) We have that

$$\begin{aligned} & |\{((\pi^2(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0} - (\pi^2(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ & + |\{((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0}\}/L_2 \oplus L_1| \\ & - |\{(\pi(L_2)^\vee - \pi^2(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ & = q^{2t_{\geq 2}(\lambda)} \{\mu(L_{(\lambda \geq 2)-2} \oplus L_{\lambda=1}) - \mu(L_{(\lambda \geq 2)-2})\mu(L_{\lambda=1})\}. \end{aligned}$$

(6) (Case 3-1) We have that

$$|\{((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus (\pi(L_1)^\vee))^{\geq 1}\}/L_2 \oplus L_1| = \mu(L_{(\lambda \geq 2)-1}) - q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}).$$

(7) (Case 5) We have that

$$|\{(((L_2)^\vee - \pi(L_2)^\vee) \oplus (L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| = \mu(L_{\lambda \geq 2} \oplus L_{\lambda=1}) - q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2} \oplus L_{\lambda=1}).$$

Here, we choose the following convention: $\forall i \geq j \geq 0$, if $L_{\lambda \geq i}$ is empty, then $\mu(L_{(\lambda \geq i)-j}) = 1$.

Proof. We only prove the cases: (Case 1-2) and (Case 2-2+Case 4-2). The other cases can be proved similarly.

In (Case 1-2), note that $\pi(L_1)^\vee = L_1$. Therefore, we have

$$|\{(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 1} - (\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 2}\}/L_2 \oplus L_1| = |\{(\pi^2(L_2)^\vee)^{\geq 1} - (\pi^2(L_2)^\vee)^{\geq 2}\}/L_2|.$$

If $L_{\lambda \geq 3}$ is empty, then we have that $\pi^2 L_2^\vee = L_2 = L_{\lambda=2}$. Therefore,

$$|\{(\pi^2(L_2)^\vee)^{\geq 1}\}/L_2| = |\{(\pi^2(L_2)^\vee)^{\geq 2}\}/L_2| = 1.$$

If $L_{\lambda \geq 3}$ is not empty, then we have

$$\begin{aligned} & |\{(\pi^2(L_2)^\vee)^{\geq 1}\}/L_2| = |\{(\pi^2 L_{\lambda \geq 3}^\vee)^{\geq 1}\}/L_{\lambda \geq 3}| = q^{2t_{\geq 3}(\lambda)} |\{(\pi(\pi^{-1} L_{\lambda \geq 3})^\vee)^{\geq 1}\}/\pi^{-1} L_{\lambda \geq 3}| \\ & = q^{2t_{\geq 3}(\lambda)} \mu^+(\pi^{-1} L_{\lambda \geq 3}) = q^{2t_{\geq 3}(\lambda)} \mu^+(L_{(\lambda \geq 3)-2}) = q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}). \end{aligned}$$

Here, we used Proposition 10.6. Also, by definition, we have that $|\{(\pi^2(L_2)^\vee)^{\geq 2}\}/L_2| = \mu^{++}(L_2) = \mu(L_{(\lambda \geq 2)-2})$. Therefore, we have that (Case 1-2) is $q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - \mu(L_{(\lambda \geq 2)-2})$.

Next, consider the case (Case 2-2+Case 4-2). First, note that

$$\begin{aligned} & |\{((\pi^2(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0} - (\pi^2(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ & + |\{((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0}\}/L_2 \oplus L_1| \\ & - |\{(\pi(L_2)^\vee - \pi^2(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ & = |\{((\pi(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0} - (\pi(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1|. \end{aligned}$$

Now, note that

$$\begin{aligned} |\{((\pi(L_2)^\vee) \oplus (L_1^\vee))^{\geq 0}\}/L_2 \oplus L_1| & = q^{2t_{\geq 2}(\lambda)} |\{((\pi^{-1} L_2)^\vee \oplus (L_1^\vee))^{\geq 0}\}/\pi^{-1} L_2 \oplus L_1| \\ & = q^{2t_{\geq 2}(\lambda)} \mu(\pi^{-1} L_2 \oplus L_1) \\ & = q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2} \oplus L_{\lambda=1}). \end{aligned}$$

Also, since $\pi L_1^\vee = L_1$, we have

$$|\{((\pi(L_2)^\vee) \oplus (\pi L_1^\vee))^{\geq 0}\}/L_2 \oplus L_1| = |\{((\pi(L_2)^\vee))^{\geq 0}\}/L_2| = q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}),$$

$$|\{(\pi(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| = q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) \times \mu(L_{\lambda=1}),$$

and

$$|\{(\pi(L_2)^\vee)^{\geq 0} \oplus (\pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| = |\{((\pi(L_2)^\vee))^{\geq 0}\}/L_2| = q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda_{\geq 2})-2}).$$

Combining these, we have that (Case 2-2+Case 4-2) is

$$q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda_{\geq 2})-2} \oplus L_{\lambda=1}) - q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda_{\geq 2})-2}) \times \mu(L_{\lambda=1}).$$

This finishes the proof of the proposition in the case (Case 2-2+Case 4-2). \square

Proposition 10.8. (cf. [LZ22b, Lemma 8.2.3], [HLSY23, Lemma 8.4]) *Assume that L (resp. M) is an O_F -lattice of rank n with hermitian matrix A_λ (resp. A_η) such that $\lambda, \eta \in \mathcal{R}_n^{0+}$, $\lambda, \eta \geq (1, \dots, 1)$. Assume further that $L \subset M \subset \pi^{-1}L$. Then, we have*

$$|((L^\vee)^{\geq 0} \setminus (M^\vee)^{\geq 0})/L| = q^{2n-1} |((\pi L^\vee)^{\geq 1} \setminus (\pi M^\vee)^{\geq 1})/L|.$$

Proof. Here, we follow the proof of [LZ22b, Lemma 8.2.3]. Consider the map

$$\begin{array}{ccc} ((L^\vee)^{\geq 0} \setminus (M^\vee)^{\geq 0})/L & \longrightarrow & ((\pi L^\vee)^{\geq 1} \setminus (\pi M^\vee)^{\geq 1})/L \\ x & \longmapsto & \pi x. \end{array}$$

To prove the proposition, it suffices to show that the above map is surjective and every fiber has size q^{2n-1} . Choose $x \in (\pi L^\vee)^{\geq 1} \setminus (\pi M^\vee)^{\geq 1}$. Then the fiber of x is given by

$$\left\{ \frac{1}{\pi}(x + y) \in (L^\vee)^{\geq 0}, y \in L/\pi L \right\}.$$

Since $x \in \pi L^\vee$, the condition $\frac{1}{\pi}(x + y) \in (L^\vee)^{\geq 0}$ is equivalent to

$$y \in L \cap \pi L^\vee / \pi L, \text{ and } (y + x, y + x) \equiv 0 \pmod{\pi^2}.$$

Choose a basis $e = \{e_1, \dots, e_n\}$ of L such that the hermitian matrix of L with respect to e is A_λ . Then, we have

$$\pi L^\vee = \bigoplus_i O_F(\pi^{-\lambda_i+1} e_i).$$

Write

$$\begin{aligned} x &= \sum_i \mu_i \pi^{-\lambda_i+1} e_i, \quad \mu_i \in O_F, \\ y &= \sum_i \nu_i e_i, \quad \nu_i \in O_F. \end{aligned}$$

Then, we have

$$\begin{aligned} (y, y) &\equiv \sum_{\lambda_i=1} (\nu_i e_i, \nu_i e_i) \equiv \sum_{\lambda_i=1} \pi \nu_i \bar{\nu}_i \pmod{\pi^2}, \\ (y, x) &= \sum_i (\nu_i e_i, \mu_i \pi^{-\lambda_i+1} e_i) = \sum_i \nu_i \bar{\mu}_i \pi. \end{aligned}$$

Since $x \in (\pi L^\vee)^{\geq 1}$, the condition $(y + x, y + x) \equiv 0 \pmod{\pi^2}$ is equivalent to

$$(10.2) \quad \frac{1}{\pi}(x, x) + \sum_{\lambda_i=1} (\nu_i \bar{\nu}_i + \nu_i \bar{\mu}_i + \bar{\nu}_i \mu_i) + \sum_{\lambda_i > 1} (\nu_i \bar{\mu}_i + \bar{\nu}_i \mu_i) \equiv 0 \pmod{\pi}.$$

Since $x \notin \pi M^\vee$, there is at least one i such that $\mu_i \not\equiv 0 \pmod{\pi}$, and $\lambda_i > 1$. Let μ_k be such a coefficient, i.e., $\mu_k \not\equiv 0 \pmod{\pi}$, and $\lambda_k > 1$.

Now, let $\beta = \frac{1}{\pi}(x, x) + \sum_{\lambda_i=1} (\nu_i \bar{\nu}_i + \nu_i \bar{\mu}_i + \bar{\nu}_i \mu_i)$. Then, (10.2) can be written as

$$\begin{aligned} & \beta + \sum_{\lambda_i > 1} (\nu_i \bar{\mu}_i + \bar{\nu}_i \mu_i) \equiv 0 \pmod{\pi} \\ \iff & \text{Tr}(\mu_k \bar{\nu}_k) \equiv -\beta - \sum_{\lambda_i > 1, i \neq k} (\nu_i \bar{\mu}_i + \bar{\nu}_i \mu_i) \pmod{\pi}. \end{aligned}$$

Choose any $\nu_i, i \neq k$, so there are $q^{2(n-1)}$ -choices. Also, $\text{Tr} : \mathbb{F}_{q^2} \rightarrow \mathbb{F}_q$ is surjective and every fiber has size q . Therefore, for each $\{\nu_i\}_{i \neq k}$, there are q -solutions of ν_k , and hence

$$|\{\frac{1}{\pi}(x+y) \in (L^\vee)^{\geq 0}, y \in L/\pi L\}| = q^{2n-1}.$$

This finishes the proof of the proposition. □

Proposition 10.9. (cf. [LZ22b, Lemma 8.2.6], [HLSY23, Lemma 8.6]) *Assume that L is an O_F -lattice of rank n and $e = \{e_1, \dots, e_n\}$ is a basis of L . Also, assume that the hermitian matrix of L with respect to e is A_λ where $\lambda \in \mathcal{R}_n^{0+}$, $\lambda \geq (1, \dots, 1)$. We choose an O_F -lattice M as follows.*

- (1) *If $\lambda_1 \geq 3$, we choose an O_F -lattice $M \supset L$ such that $M = O_F(\frac{1}{\pi}e_1) \oplus \bigoplus_{i \neq 1} O_F(e_i)$ with fundamental invariants $(\lambda_1 - 2, \lambda_2, \dots, \lambda_n)$.*
- (2) *If $\lambda_1 = \lambda_2 = 2$ (so, $\lambda_i \leq 2$ for all i), we choose an O_F -lattice $M \supset L$ such that the fundamental invariants of M are $(\lambda_1 - 1, \lambda_2 - 1, \lambda_3, \dots, \lambda_n)$.*

Then, we have that

$$|((\pi L^\vee)^{\geq 0} \setminus (\pi M^\vee)^{\geq 0})/L| = q |((\pi L^\vee)^{\geq 1} \setminus (\pi M^\vee)^{\geq 1})/L|.$$

Proof. Here, we follow the proof of [LZ22b, Lemma 8.2.6].

- (1) In this case, we have that

$$\pi L^\vee = \bigoplus_i O_F(\pi^{-\lambda_i+1} e_i),$$

and

$$\pi M^\vee = O_F(\pi^{-\lambda_1+2} e_1) \oplus \left(\bigoplus_{i \neq 1} O_F(\pi^{-\lambda_i+1} e_i) \right).$$

Fix an element $x_0 = \sum_{i \neq 1} \mu_i \pi^{-\lambda_i+1} e_i, \mu_i \in O_F$. Consider the sets

$$\begin{aligned} S_{x_0}^{\geq 0} &:= \{x \in (\pi L^\vee)^{\geq 0} \setminus (\pi M^\vee)^{\geq 0} \mid x = x_0 + \mu_1 \pi^{-\lambda_1+1} e_1, \mu_1 \in O_F\}/L \\ S_{x_0}^{\geq 1} &:= \{x \in (\pi L^\vee)^{\geq 1} \setminus (\pi M^\vee)^{\geq 1} \mid x = x_0 + \mu_1 \pi^{-\lambda_1+1} e_1, \mu_1 \in O_F\}/L. \end{aligned}$$

Then, it suffices to show that $|S_{x_0}^{\geq 0}| = q |S_{x_0}^{\geq 1}|$.

Note that $x \notin \pi M^\vee$ if and only if $\mu_1 \in O_F^\times$. Also, we have

$$(x, x) = (x_0, x_0) + \mu_1 \bar{\mu}_1 \pi^{-\lambda_1+2}.$$

Therefore, we have that

$$\begin{aligned} x \in S_{x_0}^{\geq 0} & \quad \text{if and only if} \quad \pi^{\lambda_1-2}(x_0, x_0) + \mu_1 \bar{\mu}_1 \equiv 0 \pmod{\pi^{\lambda_1-2}}, \mu_1 \in O_F^\times, \\ x \in S_{x_0}^{\geq 1} & \quad \text{if and only if} \quad \pi^{\lambda_1-2}(x_0, x_0) + \mu_1 \bar{\mu}_1 \equiv 0 \pmod{\pi^{\lambda_1-1}}, \mu_1 \in O_F^\times. \end{aligned}$$

Let us write

$$\begin{aligned}\mu_1 &= b_0 + b_1\pi + b_2\pi^2 + \dots, \\ -\pi^{\lambda_1-2}(x_0, x_0) &= c_0 + c_1\pi + c_2\pi^2 + \dots.\end{aligned}$$

Then $x \in S_{x_0}^{\geq 0}$ if and only if $b_0 \in O_F^\times$ and

$$(10.3) \quad \begin{aligned}c_0 &= b_0\bar{b}_0, \\ c_1 &= b_0\bar{b}_1 + b_1\bar{b}_0, \\ &\vdots \\ c_{\lambda_1-3} &= b_0\bar{b}_{\lambda_1-3} + \dots + b_{\lambda_1-3}\bar{b}_0.\end{aligned}$$

Also, $x \in S_{x_0}^{\geq 1}$ if and only if $x \in S_{x_0}^{\geq 0}$ and

$$(10.4) \quad \begin{aligned}c_{\lambda_1-2} &= b_0\bar{b}_{\lambda_1-2} + \dots + b_{\lambda_1-2}\bar{b}_0 \\ \iff \operatorname{Tr}(b_0\bar{b}_{\lambda_1-2}) &= c_{\lambda_1-2} - b_1\bar{b}_{\lambda_1-3} - \dots - b_{\lambda_1-3}\bar{b}_1.\end{aligned}$$

Now, for each $\{b_0, b_1, \dots, b_{\lambda_1-3}\}$ satisfying (10.3), there are q^2 -choices of b_{λ_1-2} so that $x \in S_{x_0}^{\geq 0}$. Also, there are q -choices of b_{λ_1-2} so that (10.4) is true, and hence $x \in S_{x_0}^{\geq 1}$. This shows that $|S_{x_0}^{\geq 0}| = q|S_{x_0}^{\geq 1}|$.

- (2) In this case, we may choose a basis $f = \{f_1, f_2, \dots, f_n\}$ of πM^\vee such that the hermitian matrix of πM^\vee with respect to f is

$$\begin{pmatrix} \pi^1 & & & & \\ & \pi^1 & & & \\ & & \pi^{-\lambda_3+2} & & \\ & & & \ddots & \\ & & & & \pi^{-\lambda_n+2} \end{pmatrix},$$

and $\pi L^\vee = O_F(\pi^{-1}(\varepsilon f_1 + f_2)) \oplus O_F(\pi^{-1}(f_1 - \bar{\varepsilon} f_2)) \oplus \bigoplus_{i \neq 1,2} O_F(f_i)$, for some ε such that $1 + \varepsilon\bar{\varepsilon} = \pi$. Then, $\{\pi^{-1}(\varepsilon f_1 + f_2), f_2, f_3, \dots, f_n\}$ forms a basis for πL^\vee .

We fix $x_0 = \sum_{i \neq 1} \mu_i f_i$, $\mu_i \in O_F$, and let $x_0 = x_1 + \mu_2 f_2$. Then, consider the sets

$$\begin{aligned}S_{x_0}^{\geq 0} &:= \{x \in (\pi L^\vee)^{\geq 0} \setminus (\pi M^\vee)^{\geq 0} \mid x = x_0 + \mu_1 \pi^{-1}(\varepsilon f_1 + f_2), \mu_1 \in O_F\} / L, \\ S_{x_0}^{\geq 1} &:= \{x \in (\pi L^\vee)^{\geq 1} \setminus (\pi M^\vee)^{\geq 1} \mid x = x_0 + \mu_1 \pi^{-1}(\varepsilon f_1 + f_2), \mu_1 \in O_F\} / L.\end{aligned}$$

Note that $x \notin \pi M^\vee$ if and only if $\mu_1 \in O_F^\times$. Also, we have that

$$\begin{aligned}(x, x) &= (x_1 + \mu_2 f_2 + \mu_1 \pi^{-1}(\varepsilon f_1 + f_2), x_1 + \mu_2 f_2 + \mu_1 \pi^{-1}(\varepsilon f_1 + f_2)) \\ &= (x_1, x_1) + \mu_2 \bar{\mu}_2 \pi + \mu_2 \bar{\mu}_1 + \bar{\mu}_2 \mu_1 + \mu_1 \bar{\mu}_1.\end{aligned}$$

Therefore, we have

$$\begin{aligned}x \in S_{x_0}^{\geq 0} &\quad \text{if and only if} \quad (x_1, x_1) + \mu_2 \bar{\mu}_2 \pi + \mu_2 \bar{\mu}_1 + \bar{\mu}_2 \mu_1 + \mu_1 \bar{\mu}_1 \in O_F, \text{ and } \mu_1 \in O_F^\times, \\ x \in S_{x_0}^{\geq 1} &\quad \text{if and only if} \quad (x_1, x_1) + \mu_2 \bar{\mu}_2 \pi + \mu_2 \bar{\mu}_1 + \bar{\mu}_2 \mu_1 + \mu_1 \bar{\mu}_1 \equiv 0 \pmod{\pi}, \text{ and } \mu_1 \in O_F^\times.\end{aligned}$$

Write

$$\begin{aligned}-(x_1, x_1) &= d_0 + d_1\pi + d_2\pi^2 + \dots, \\ \mu_1 &= b_0 + b_1\pi + b_2\pi^2 + \dots,\end{aligned}$$

$$\mu_2 = c_0 + c_1\pi + c_2\pi^2 + \dots$$

Then, $x \in S_{x_0}^{\geq 0}$ if and only if $b_0 \in O_F^\times$. Therefore, there are q^2 -choices of c_0 .

Also, $x \in S_{x_0}^{\geq 1}$ if and only if $b_0 \in O_F^\times$ and

$$\begin{aligned} d_0 &= c_0\bar{b}_0 + \bar{c}_0b_0 + b_0\bar{b}_0 \\ \iff \text{Tr}(c_0\bar{b}_0) &= d_0 - b_0\bar{b}_0. \end{aligned}$$

Therefore, for each $b_0 \in O_F^\times$, there are q -choices of c_0 . This shows that $|S_{x_0}^{\geq 0}| = q|S_{x_0}^{\geq 1}|$. This finishes the proof of the proposition. □

Proposition 10.10. *Assume that $\lambda \in \mathcal{R}_n^{0+}$.*

(1) *If $\lambda_1 \geq 3$, we define*

$$\eta = (\lambda_1 - 2, \lambda_2, \dots, \lambda_n) \in \mathcal{R}_n^{0+},$$

(if necessary, we change the order of η_i 's so that $\eta \in \mathcal{R}_n^{0+}$).

Then, we have

$$\begin{aligned} \mu(L_\lambda) &= \mu(L_{\lambda \geq 1}) \\ &= q^2\mu(L_{\eta \geq 1}) + q^{2t_{\geq 1}(\lambda) + 2t_{\geq 2}(\lambda) - 2}\mu(L_{(\lambda \geq 2) - 2}) - q^{2t_{\geq 1}(\lambda) + 2t_{\geq 2}(\lambda)}\mu(L_{(\eta \geq 2) - 2}). \end{aligned}$$

(2) *If $\lambda_1 = \lambda_2 = 2$, we define*

$$\eta = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_n) \in \mathcal{R}_n^{0+},$$

(if necessary, we change the order of η_i 's so that $\eta \in \mathcal{R}_n^{0+}$).

Then, we have

$$\begin{aligned} \mu(L_\lambda) &= \mu(L_{\lambda \geq 1}) \\ &= q^2\mu(L_{\eta \geq 1}) + q^{2t_{\geq 1}(\lambda) + 2t_{\geq 2}(\lambda) - 2}\mu(L_{(\lambda \geq 2) - 2}) - q^{2t_{\geq 1}(\eta) + 2t_{\geq 2}(\eta)}\mu(L_{(\eta \geq 2) - 2}). \end{aligned}$$

Here, we choose the following convention: If $L_{\lambda \geq 2}$ is empty, we assume $\mu(L_{(\lambda \geq 2) - 2}) = 1$.

Proof. By Proposition 10.8 and Proposition 10.9, we have

$$|((L_\lambda^\vee)^{\geq 0} \setminus (L_\eta^\vee)^{\geq 0}) / L_\lambda| = q^{2t_{\geq 1}(\lambda) - 1} |((\pi L_\lambda^\vee)^{\geq 1} \setminus (\pi L_\eta^\vee)^{\geq 1}) / L_\lambda|,$$

and

$$|((\pi L_\lambda^\vee)^{\geq 0} \setminus (\pi L_\eta^\vee)^{\geq 0}) / L_\lambda| = q |((\pi L_\lambda^\vee)^{\geq 1} \setminus (\pi L_\eta^\vee)^{\geq 1}) / L_\lambda|.$$

Therefore, we have

$$(10.5) \quad |((L_\lambda^\vee)^{\geq 0} \setminus (L_\eta^\vee)^{\geq 0}) / L_\lambda| = q^{2t_{\geq 1}(\lambda) - 2} |((\pi L_\lambda^\vee)^{\geq 0} \setminus (\pi L_\eta^\vee)^{\geq 0}) / L_\lambda|.$$

Note that

$$\begin{aligned} |(L_\lambda^\vee)^{\geq 0} / L_\lambda| &= \mu(L_\lambda), \\ |(L_\eta^\vee)^{\geq 0} / L_\lambda| &= q^2 |(L_\eta^\vee)^{\geq 0} / L_\eta| = q^2 \mu(L_\eta). \end{aligned}$$

Also, note that $(\pi L_\lambda^\vee)^{\geq 0}/L_\lambda = (\pi(L_{\lambda \geq 2})^\vee)^{\geq 0}/L_{\lambda \geq 2}$, and it is trivial if $L_{\lambda \geq 2}$ is empty. Since the fundamental invariants of $L_{\lambda \geq 2}$ are at least 2, any element x of $\pi^{-1}L_{\lambda \geq 2}$ has valuation at least 0. Therefore, we have

$$\begin{aligned} |(\pi L_\lambda^\vee)^{\geq 0}/L_\lambda| &= |(\pi(L_{\lambda \geq 2})^\vee)^{\geq 0}/L_{\lambda \geq 2}| = q^{2t_{\geq 2}(\lambda)} |((\pi^{-1}L_{\lambda \geq 2})^\vee)^{\geq 0}/(\pi^{-1}L_{\lambda \geq 2})| \\ &= q^{2t_{\geq 2}(\lambda)} \mu(\pi^{-1}L_{\lambda \geq 2}) = q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}). \end{aligned}$$

Similarly, we have

$$|(\pi L_\eta^\vee)^{\geq 0}/L_\eta| = q^2 |(\pi L_\eta^\vee)^{\geq 0}/L_\eta| = q^{2+2t_{\geq 2}(\eta)} \mu(L_{(\eta \geq 2)-2}).$$

Now, by (10.5) and the fact that $t_{\geq 1}(\lambda) = t_{\geq 1}(\eta)$, we have the proof of the proposition. \square

Lemma 10.11. *For $\lambda \in \mathcal{R}_n^{0+}$, $\lambda \geq (1, 1, \dots, 1)$, we have that*

$$\mu(L_\lambda) = q^{2n-1} \mu(L_{\lambda-1}) - (-q)^{|\lambda|-1} (q-1).$$

Proof. First, consider the case $\lambda = (1, 1, \dots, 1)$. Then, it is easy to see that (see [VW11, Example 5.6], for example)

$$\begin{aligned} \mu(L_{1,\dots,1}) &= |\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_1^{q+1} + \dots + x_n^{q+1} = 0\}| \\ &= q^{2n-1} + (-q)^n + (-q)^{n-1} = q^{2n-1} \mu(L_{0,\dots,0}) - (-q)^{n-1} (q-1). \end{aligned}$$

Similarly, if $\lambda = (2, \overbrace{1, \dots, 1}^{n-1})$, we have that

$$\begin{aligned} \mu(L_\lambda) &= |((L_{2,1,\dots,1})^\vee)^{\geq 0}/L_{2,1,\dots,1}| = q^2 \mu(\underbrace{L_{1,\dots,1}}_{n-1}) = q^{2n-1} + (-q)^{n+1} + (-q)^n \\ &= q^{2n-1} \mu(L_1) - (-q)^n (q-1). \end{aligned}$$

Now, we will prove the lemma by induction on $|\lambda|$.

Assume that $\lambda = (\overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$, $a \geq 2$, and let $\eta = (\overbrace{2, \dots, 2}^{a-2}, \overbrace{1, \dots, 1}^{b+2})$. Then, by Proposition 10.10 (2), we have

$$\mu(L_\lambda) = q^2 \mu(L_\eta) + q^{2n+2a-2} - q^{2n+2a-4}.$$

Also, we have

$$\begin{aligned} \mu(L_{\lambda-1}) &= \mu(\underbrace{L_{1,\dots,1}}_a) = q^{2a-1} + (-q)^a + (-q)^{a-1}, \\ q^2 \mu(L_{\eta-1}) &= q^2 \mu(\underbrace{L_{1,\dots,1}}_{a-2}) = q^{2a-3} + (-q)^a + (-q)^{a-1}, \\ \mu(L_{\lambda-1}) &= q^2 \mu(L_{\eta-1}) + q^{2a-1} - q^{2a-3}. \end{aligned}$$

Therefore, we have that

$$\mu(L_\lambda) - q^{2n-1} \mu(L_{\lambda-1}) = q^2 \{\mu(L_\eta) - q^{2n-1} \mu(L_{\eta-1})\}.$$

Now, by our induction hypothesis, we have that

$$q^2 \{\mu(L_\eta) - q^{2n-1} \mu(L_{\eta-1})\} = -(-q)^{|\eta|-1+2} (q-1) = -(-q)^{|\lambda|-1} (q-1).$$

This finishes the proof of the lemma when $\lambda = (\overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$, $a \geq 2$.

Now, assume that $\lambda = (\overbrace{3, \dots, 3}^{n-a-b}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$, and let $\eta = (\overbrace{3, \dots, 3}^{n-a-b-1}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^{b+1})$. Then, by Proposition 10.10 (1), we have

$$\mu(L_\lambda) = q^2 \mu(L_\eta) + q^{4n-2b-2} \mu(L_{(\lambda \geq 2)_-2}) - q^{4n-2b-2} \mu(L_{(\eta \geq 2)_-2}).$$

Also, by the previous case, we have

$$\begin{aligned} \mu(L_{\lambda-1}) &= q^{2n-2b-1} \mu(L_{(\lambda \geq 2)_-2}) - (-q)^{|\lambda|-n-1} (q-1), \\ \mu(L_{\eta-1}) &= q^{2n-2b-3} \mu(L_{(\eta \geq 2)_-2}) - (-q)^{|\eta|-n-1} (q-1), \\ \mu(L_{\lambda-1}) &= q^2 \mu(L_{\eta-1}) + q^{2n-2b-1} \mu(L_{(\lambda \geq 2)_-2}) - q^{2n-2b-1} \mu(L_{(\eta \geq 2)_-2}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mu(L_\lambda) - q^{2n-1} \mu(L_{\lambda-1}) &= q^2 \mu(L_\eta) + q^{4n-2b-2} \mu(L_{(\lambda \geq 2)_-2}) - q^{4n-2b-2} \mu(L_{(\eta \geq 2)_-2}) \\ &\quad - q^{2n-1} \{q^2 \mu(L_{\eta-1}) + q^{2n-2b-1} \mu(L_{(\lambda \geq 2)_-2}) - q^{2n-2b-1} \mu(L_{(\eta \geq 2)_-2})\} \\ &= q^2 \{\mu(L_\eta) - q^{2n-1} \mu(L_{\eta-1})\} \\ &= -(-q)^{|\eta|+2-1} (q-1) \text{ (by our inductive hypothesis)} \\ &= -(-q)^{|\lambda|-1} (q-1). \end{aligned}$$

This finishes the proof of the lemma when $\lambda = (\overbrace{3, \dots, 3}^{n-a-b}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$.

Now, assume that $\lambda = (\lambda_1 = 4, \lambda_2, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$, $\lambda_{n-a-b} \geq 3$. Let

$$\eta = (\lambda_2, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^{a+1}, \overbrace{1, \dots, 1}^b).$$

Then, by Proposition 10.10 (1), we have

$$\mu(L_\lambda) = q^2 \mu(L_\eta) + q^{4n-2b-2} \mu(L_{(\lambda \geq 2)_-2}) - q^{4n-2b} \mu(L_{(\eta \geq 2)_-2}).$$

Similarly, we have $t_{\geq 2}(\lambda-1) = n-a-b$, $t_{\geq 2}(\eta-1) = n-a-b-1$, $t_{\geq 1}(\lambda-1) = t_{\geq 1}(\eta-1) = n-b$, and

$$\mu(L_{\lambda-1}) = q^2 \mu(L_{\eta-1}) + q^{4n-2a-4b-2} \mu(L_{(\lambda \geq 3)_-3}) - q^{4n-2a-4b-2} \mu(L_{(\eta \geq 3)_-3}).$$

Therefore,

$$\begin{aligned} \mu(L_\lambda) - q^{2n-1} \mu(L_{\lambda-1}) &= q^2 \{\mu(L_\eta) - q^{2n-1} \mu(L_{\eta-1})\} \\ &\quad + q^{4n-2b-2} \{\mu(L_{(\lambda \geq 2)_-2}) - q^{2n-2a-2b-1} \mu(L_{(\lambda \geq 3)_-3})\} \\ &\quad - q^{4n-2b} \{\mu(L_{(\eta \geq 2)_-2}) - q^{2n-2a-2b-3} \mu(L_{(\eta \geq 3)_-3})\} \\ \text{(by our inductive hypothesis)} &= -(-q)^{|\eta|+1} (q-1) + q^{4n-2b-2} (-(-q)^{|\lambda|-b-2(n-b)-1} (q-1)) \\ &\quad - (-q)^{4n-2b} (-(-q)^{|\eta|-b-2(n-b)-1} (q-1)) \\ &= -(-q)^{|\lambda|-1} (q-1). \end{aligned}$$

This finishes the proof of the lemma when $\lambda = (\lambda_1 = 4, \lambda_2, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$, $\lambda_{n-a-b} \geq 3$.

Finally, assume that $\lambda = (\lambda_1, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^a, \overbrace{1, \dots, 1}^b)$, $\lambda_1 \geq 5$. In this case, let $\eta = (\lambda_1 - 2, \lambda_2, \dots, \lambda_n)$ (if necessary, we change the order of η_i 's so that $\eta \in \mathcal{R}_n^{0+}$). Then, by Proposition 10.10 (1), we have

$$\mu(L_\lambda) = q^2 \mu(L_\eta) + q^{4n-2b-2} \mu(L_{(\lambda \geq 2)-2}) - q^{4n-2b} \mu(L_{(\eta \geq 2)-2}),$$

and

$$\mu(L_{\lambda-1}) = q^2 \mu(L_{\eta-1}) + q^{4n-2a-4b-2} \mu(L_{(\lambda \geq 3)-3}) - q^{4n-2a-4b} \mu(L_{(\eta \geq 3)-3}).$$

Therefore,

$$\begin{aligned} \mu(L_\lambda) - q^{2n-1} \mu(L_{\lambda-1}) &= q^2 \{ \mu(L_\eta) - q^{2n-1} \mu(L_{\eta-1}) \} \\ &\quad + q^{4n-2b-2} \{ \mu(L_{(\lambda \geq 2)-2}) - q^{2n-2a-2b-1} \mu(L_{(\lambda \geq 3)-3}) \} \\ &\quad - q^{4n-2b} \{ \mu(L_{(\eta \geq 2)-2}) - q^{2n-2a-2b-1} \mu(L_{(\eta \geq 3)-3}) \} \\ \text{(by our inductive hypothesis)} &= -(-q)^{|\eta|+1} (q-1) + q^{4n-2b-2} (-(-q)^{|\lambda|-b-2(n-b)-1} (q-1)) \\ &\quad - (-q)^{4n-2b} (-(-q)^{|\eta|-b-2(n-b)-1} (q-1)) \\ &= -(-q)^{|\lambda|-1} (q-1). \end{aligned}$$

This finishes the proof of the lemma. □

Lemma 10.12. *Assume that $a \geq 1$, and $j \in \mathbb{Z}$. Let $\kappa_{a,i}$ be the constants such that*

$$(1-X)(1-(-q)X) \dots (1-(-q)^{a-2}X) = \sum_{i=0}^{a-1} \kappa_{a,i} X^i.$$

Then, we have

$$D_{n,h}(a, b, c) = \sum_{i=0}^{a-1} \kappa_{a,i} (-q)^{i(a+b-h+1)} D_{n-i, h-i}(1, b+a-1-i, c).$$

Proof. This follows from Theorem 9.4. □

Definition 10.13. Assume that $a+b+c = n-1$.

(1) We define the function $E_{n,h}(a, b, c)$ by

$$\begin{aligned} E_{n,h}(a, b, c) := & \begin{aligned} & D_{n,h}(a+1, b, c) \\ & + ((-q)^a - 1) \times D_{n,h}(a, b+1, c) \\ & + (-q)^a ((-q)^a - 1) \times D_{n,h}(a, b, c+1) \\ & - (-q)^{2a} (1 - (-q)^b) (1 - (-q)^{b-1}) \times D_{n,h}(a+1, b-2, c+2) \\ & - (-q)^{2a+b-1} (1 - (-q)^b) (1 - (-q)^a) \times D_{n,h}(a, b-1, c+2). \end{aligned} \end{aligned}$$

(2) We define the function $F_{n,h}(a, b, c)$ by

$$\begin{aligned} F_{n,h}(a, b, c) := & \begin{aligned} & D_{n,h}(a, b+1, c) \\ & + ((-q)^{a-1} - 1) \times D_{n,h}(a-1, b+2, c) \\ & - (-q)^{a-1} \times D_{n,h}(a, b, c+1) \\ & + (-q)^{2a+b-1} (1 - (-q)^b) \times D_{n,h}(a, b-1, c+2) \\ & + (-q)^{2a+2b-1} (1 - (-q)^{a-1}) \times D_{n,h}(a-1, b, c+2) \end{aligned} \end{aligned}$$

Lemma 10.14. *Assume that $a \geq 1$. Then, we have*

$$E_{n,h}(a, b, c) = 0.$$

Proof. To simplify notation, we assume that for $k \geq 1$,

$$\begin{aligned} X_0^k &:= D_{n-k+1, h-k+1}(1, b+a-k+1, c), \\ X_1^k &:= D_{n-k+1, h-k+1}(1, b+a-k, c+1), \\ X_2^k &:= D_{n-k+1, h-k+1}(1, b+a-k-1, c+2). \end{aligned}$$

Then, by Lemma 10.12, we have that

$$\begin{aligned} D_{n,h}(a+1, b, c) &= X_0(1 - (-q)^{a+b-h+2}X_0) \dots (1 - (-q)^{2a+b-h+1}X_0), \\ D_{n,h}(a, b+1, c) &= X_0(1 - (-q)^{a+b-h+2}X_0) \dots (1 - (-q)^{2a+b-h}X_0), \\ D_{n,h}(a, b, c+1) &= X_1(1 - (-q)^{a+b-h+1}X_1) \dots (1 - (-q)^{2a+b-h-1}X_1), \\ D_{n,h}(a+1, b-2, c+2) &= X_2(1 - (-q)^{a+b-h}X_2) \dots (1 - (-q)^{2a+b-h-1}X_2), \\ D_{n,h}(a, b-1, c+2) &= X_2(1 - (-q)^{a+b-h}X_2) \dots (1 - (-q)^{2a+b-h-2}X_2). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} &D_{n,h}(a+1, b, c) + ((-q)^a - 1)D_{n,h}(a, b+1, c) \\ &= (-q)^a X_0(1 - (-q)^{a+b-h+1}X_0) \dots (1 - (-q)^{2a+b-h}X_0) \\ &= (-q)^a X_0(1 - (-q)^{2a+b-h}X_0) \left(\sum_{t=0}^{a-1} \kappa_{a,t} (-q)^{t(a+b-h+1)} X_0^t \right). \end{aligned}$$

This implies that for $1 \leq t \leq a+1$, X_0^t -terms are

$$(10.6) \quad \begin{cases} (-q)^a \{ \kappa_{a,t-1} (-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h} \kappa_{a,t-2} (-q)^{(a+b-h+1)(t-2)} \} X_0^t & \text{if } t \neq a+1, \\ -(-q)^{3a+b-h} \kappa_{a,a-1} (-q)^{(a+b-h+1)(a-1)} X_0^{a+1} & \text{if } t = a+1. \end{cases}$$

Similarly, for $1 \leq t \leq a$, X_1^t -terms in $(-q)^a((-q)^a - 1)D_{n,h}(a, b, c+1)$ are

$$(10.7) \quad (-q)^a((-q)^a - 1) \kappa_{a,t-1} (-q)^{(a+b-h+1)(t-1)} X_1^t.$$

Finally, for $1 \leq t \leq a+1$, X_2^t -terms in

$$\begin{aligned} &-(-q)^{2a}(1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(a+1, b-2, c+2) \\ &-(-q)^{2a+b-1}(1 - (-q)^b)(1 - (-q)^a)D_{n,h}(a, b-1, c+2) \end{aligned}$$

are

$$(10.8) \quad \begin{cases} \{ -\kappa_{a,t-1} (-q)^{(a+b-h)(t-1)+2a}(1 - (-q)^b)(1 - (-q)^{a+b-1}) \\ + \kappa_{a,t-2} (-q)^{(a+b-h)(t-2)+4a+b-h-1}(1 - (-q)^b)(1 - (-q)^{b-1}) \} X_2^t & \text{if } t \neq a+1, \\ \kappa_{a,a-1} (-q)^{(a+b-h)(a-1)+4a+b-h-1}(1 - (-q)^b)(1 - (-q)^{b-1}) X_2^{a+1} & \text{if } t = a+1. \end{cases}$$

Note that if $a+b < h-1$, then X_0^t, X_1^t , and X_2^t are all zero. Therefore, all of the degree t -terms of $E_{n,h}(a, b, c)$ are 0, and hence $E_{n,h}(a, b, c) = 0$ in this case.

Now, assume that $a+b \geq h+2$. Then, by Theorem 9.8, we have that

$$(10.9) \quad \begin{aligned} X_0^t &= \prod_{l=h-t+2}^{a+b-t+1} (1 - (-q)^l), \\ X_1^t &= \prod_{l=h-t+2}^{a+b-t} (1 - (-q)^l), \\ X_2^t &= \prod_{l=h-t+2}^{a+b-t-1} (1 - (-q)^l). \end{aligned}$$

Combining (10.6), (10.7), (10.8), and (10.9), we can see that the degree t -terms of $E_{n,h}(a, b, c)$ can be written as follows: if $1 \leq t \leq a$, we have

$$\begin{aligned}
(10.10) \quad & \prod_{l=h-t+2}^{a+b-t-1} (1 - (-q)^l) \left\{ (-q)^a \{ \kappa_{a,t-1}(-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h} \kappa_{a,t-2}(-q)^{(a+b-h+1)(t-2)} \} \right. \\
& \times (1 - (-q)^{a+b-t+1})(1 - (-q)^{a+b-t}) \\
& + (-q)^a ((-q)^a - 1) \kappa_{a,t-1}(-q)^{(a+b-h+1)(t-1)} (1 - (-q)^{a+b-t}) \\
& + \{ -\kappa_{a,t-1}(-q)^{(a+b-h)(t-1)+2a} (1 - (-q)^b) (1 - (-q)^{a+b-1}) \\
& + \kappa_{a,t-2}(-q)^{(a+b-h)(t-2)+4a+b-h-1} (1 - (-q)^b) (1 - (-q)^{b-1}) \} \\
& = ((-q)^{a+2b} - (-q)^t) (-q)^{(a+b-h)(t-2)+3a+b-h-t-2} \\
& \{ -((-q)^{t+1} - (-q)^2) \kappa_{a,t-1} + ((-q)^t - (-q)^{a+1}) \kappa_{a,t-2} \}.
\end{aligned}$$

Now, we claim that

$$(10.11) \quad \{ -((-q)^{t+1} - (-q)^2) \kappa_{a,t-1} + ((-q)^t - (-q)^{a+1}) \kappa_{a,t-2} \} = 0.$$

Recall from Lemma 10.12 that

$$\begin{aligned}
(1 - X)(1 - (-q)X) \dots (1 - (-q)^{a-2}X) &= \sum_{t=0}^{a-1} \kappa_{a,t} X^t, \\
(1 - (-q)X)(1 - (-q)^2X) \dots (1 - (-q)^{a-1}X) &= \sum_{t=0}^{a-1} \kappa_{a,t} (-q)^t X^t.
\end{aligned}$$

Therefore,

$$(1 - (-q)^{a-1}X) \sum_{t=0}^{a-1} \kappa_{a,t} X^t = (1 - X) \sum_{t=0}^{a-1} \kappa_{a,t} (-q)^t X^t.$$

By comparing degree $t - 1$ terms, we have that

$$\begin{aligned}
(10.12) \quad & \kappa_{a,t-1} - (-q)^{a-1} \kappa_{a,t-2} = \kappa_{a,t-1} (-q)^{t-1} - \kappa_{a,t-2} (-q)^{t-2} \\
& \iff -((-q)^{t-1} - 1) \kappa_{a,t-1} + ((-q)^{t-2} - (-q)^{a-1}) \kappa_{a,t-2} = 0.
\end{aligned}$$

Therefore, we have that (10.11) holds, and hence (10.10) is zero. This shows that the degree t -term of $E_{n,h}(a, b, c)$ is zero for $1 \leq t \leq a$.

For $t = a + 1$, we have that the degree $a + 1$ -terms of $E_{n,h}(a, b, c)$ are

$$\begin{aligned}
& \prod_{l=h-a+1}^{b-2} (1 - (-q)^l) \left\{ -(-q)^{3a+b-h} \kappa_{a,a-1} (-q)^{(a+b-h+1)(a-1)} (1 - (-q)^b) (1 - (-q)^{b-1}) \right. \\
& \quad \left. + \kappa_{a,a-1} (-q)^{(a+b-h)(a-1)+4a+b-h-1} (1 - (-q)^b) (1 - (-q)^{b-1}) \right\} \\
& = 0.
\end{aligned}$$

This shows that $E_{n,h}(a, b, c) = 0$ holds when $a + b \geq h + 2$.

Now, the remaining cases are $a + b = h - 1$, h , and $h + 1$.

When $a + b = h - 1$, we have that $X_0^t = 1$, $X_1^t = X_2^t = 0$ by Theorem 9.8. Therefore, we have that

$$\begin{aligned}
& D_{n,h}(a + 1, b, c) + ((-q)^a - 1) D_{n,h}(a, b + 1, c) \\
& = (-q)^a (1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h}) \\
& = 0 \quad (\text{since } a + b - h + 1 = 0).
\end{aligned}$$

Also, we have $D_{n,h}(a, b, c + 1) = 0$, $D_{n,h}(a + 1, b - 2, c + 2) = 0$, and $D_{n,h}(a, b - 1, c + 2) = 0$. This shows that $E_{n,h}(a, b, c) = 0$ holds when $a + b = h - 1$.

When $a + b = h$, we have that $X_0^t = X_1^t = 1$ and $X_2^t = 0$ by Theorem 9.8. Therefore, we have that

$$\begin{aligned} D_{n,h}(a + 1, b, c) + ((-q)^a - 1)D_{n,h}(a, b + 1, c) &= (-q)^a(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h}); \\ (-q)^a((-q)^a - 1)D_{n,h}(a, b, c + 1) &= (-q)^a((-q)^a - 1)(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h-1}); \\ -(-q)^{2a}(1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(a + 1, b - 2, c + 2) &= 0; \\ -(-q)^{2a+b-1}(1 - (-q)^b)(1 - (-q)^a)D_{n,h}(a, b - 1, c + 2) &= 0. \end{aligned}$$

Therefore, $E_{n,h}(a, b, c)$ is

$$\begin{aligned} &(-q)^a(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h}) + (-q)^a((-q)^a - 1)(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h-1}) \\ &= (-q)^a(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h-1})\{(-q)^a - (-q)^{2a+b-h}\} \\ &= 0 \quad (\text{since } 2a + b - h = a). \end{aligned}$$

Now, assume that $a + b = h + 1$. In this case, by Theorem 9.8, we have that $X_0^t = (1 - (-q)^{h-t+2})$, $X_1^t = 1$, and $X_2^t = 1$. Therefore, we have

$$\begin{aligned} (10.13) \quad &D_{n,h}(a + 1, b, c) + ((-q)^a - 1)D_{n,h}(a, b + 1, c) \\ &= (-q)^a(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h}) \\ &\quad - (-q)^a(-q)^{h+1}(1 - (-q)^{a+b-h+1}(-q)^{-1}) \dots (1 - (-q)^{2a+b-h}(-q)^{-1}) \\ &= (-q)^a(1 - (-q)^2) \dots (1 - (-q)^a)(1 - (-q)^{a+1} - (-q)^{h+1} + (-q)^{h+2}). \end{aligned}$$

Also, we have

$$(-q)^a((-q)^a - 1)D_{n,h}(a, b, c + 1) = (-q)^a((-q)^a - 1)(1 - (-q)^2) \dots (1 - (-q)^a),$$

and

$$\begin{aligned} &-(-q)^{2a}(1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(a + 1, b - 2, c + 2) \\ &-(-q)^{2a+b-1}(1 - (-q)^b)(1 - (-q)^a)D_{n,h}(a, b - 1, c + 2) \\ &= -(-q)^{2a}(1 - (-q)^b)(1 - (-q)^{b-1})(1 - (-q)^{a+b-h}) \dots (1 - (-q)^{2a+b-h-1}) \\ &\quad - (-q)^{2a+b-1}(1 - (-q)^b)(1 - (-q)^a)(1 - (-q)^{a+b-h}) \dots (1 - (-q)^{2a+b-h-2}) \\ &= -(-q)^{2a}(1 - (-q)^b)(1 - (-q)^1) \dots (1 - (-q)^a). \end{aligned}$$

Therefore, $E_{n,h}(a, b, c)$ is

$$\begin{aligned} &(-q)^a(1 - (-q)^2) \dots (1 - (-q)^a)(1 - (-q)^{a+1} - (-q)^{h+1} + (-q)^{h+2}) \\ &+ (-q)^a((-q)^a - 1)(1 - (-q)^2) \dots (1 - (-q)^a) - (-q)^{2a}(1 - (-q)^b)(1 - (-q)^1) \dots (1 - (-q)^a) \\ &= 0. \end{aligned}$$

Hence, $E_{n,h}(a, b, c) = 0$ holds when $a + b = h + 1$.

This finishes the proof of the lemma. □

Lemma 10.15. *Assume that $a = 0$. Then, we have*

$$E_{n,h}(a, b, c) = \begin{cases} 0 & \text{if } b \geq h + 2 \text{ or } b \leq h - 2, \\ 1 & \text{if } b = h - 1, h, \\ q^{2h+1} + (-q)^h & \text{if } b = h + 1. \end{cases}$$

Proof. Note that when $a = 0$, we have that $E_{n,h}(a, b, c)$ is

$$D_{n,h}(1, b, c) - (1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(1, b - 2, c + 2).$$

If $b \leq h - 2$, then we have that both $D_{n,h}(1, b, c)$ and $D_{n,h}(1, b - 2, c + 2)$ are zero.

If $b \geq h + 2$, then by Theorem 9.8, we have that

$$\begin{aligned} & D_{n,h}(1, b, c) - (1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(1, b - 2, c + 2) \\ &= \prod_{l=h+1}^b (1 - (-q)^l) - (1 - (-q)^b)(1 - (-q)^{b-1}) \prod_{l=h+1}^{b-2} (1 - (-q)^l) = 0. \end{aligned}$$

Now, assume that $b = h + 1$. Then, by Theorem 9.8, we have

$$\begin{aligned} & D_{n,h}(1, b, c) - (1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(1, b - 2, c + 2) \\ &= (1 - (-q)^{h+1}) - (1 - (-q)^{h+1})(1 - (-q)^h) = (-q)^h(1 - (-q)^{h+1}) = q^{2h+1} + (-q)^h. \end{aligned}$$

Finally, if $b = h, h - 1$, then by Theorem 9.8, we have

$$D_{n,h}(1, b, c) - (1 - (-q)^b)(1 - (-q)^{b-1})D_{n,h}(1, b - 2, c + 2) = 1.$$

This finishes the proof of the lemma. □

Lemma 10.16. *Assume that $a \geq 2$. Then, we have*

$$F_{n,h}(a, b, c) = 0$$

Proof. To simplify notation, we follow the notation in the proof of Lemma 10.14. Then, by Lemma 10.12, we have

$$\begin{aligned} D_{n,h}(a, b + 1, c) &= X_0(1 - (-q)^{a+b-h+2}X_0) \dots (1 - (-q)^{2a+b-h}X_0), \\ D_{n,h}(a - 1, b + 2, c) &= X_0(1 - (-q)^{a+b-h+2}X_0) \dots (1 - (-q)^{2a+b-h-1}X_0), \\ D_{n,h}(a, b, c + 1) &= X_1(1 - (-q)^{a+b-h+1}X_1) \dots (1 - (-q)^{2a+b-h-1}X_1), \\ D_{n,h}(a, b - 1, c + 2) &= X_2(1 - (-q)^{a+b-h}X_2) \dots (1 - (-q)^{2a+b-h-2}X_2), \\ D_{n,h}(a - 1, b, c + 2) &= X_2(1 - (-q)^{a+b-h}X_2) \dots (1 - (-q)^{2a+b-h-3}X_2). \end{aligned}$$

This implies that for $1 \leq t \leq a$, X_0^t -terms of $D_{n,h}(a, b + 1, c) + ((-q)^{a-1} - 1)D_{n,h}(a - 1, b + 2, c)$ are

$$(10.14) \quad \begin{cases} (-q)^{a-1} \{ \kappa_{a-1, t-1} (-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h-1} \kappa_{a-1, t-2} (-q)^{(a+b-h+1)(t-2)} \} X_0^t, & \text{if } t \neq a, \\ -(-q)^{3a+b-h-2} \kappa_{a-1, a-2} (-q)^{(a+b-h+1)(a-2)} X_0^a & \text{if } t = a. \end{cases}$$

Similarly, for $1 \leq t \leq a$, X_1^t -terms in $-(-q)^{a-1}D_{n,h}(a, b, c + 1)$ are

$$(10.15) \quad \begin{cases} -(-q)^{a-1} \{ \kappa_{a-1, t-1} (-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h-1} \kappa_{a-1, t-2} (-q)^{(a+b-h+1)(t-2)} \} X_1^t, & \text{if } t \neq a, \\ (-q)^{3a+b-h-2} \kappa_{a-1, a-2} (-q)^{(a+b-h+1)(a-2)} X_1^a & \text{if } t = a. \end{cases}$$

Finally, for $1 \leq t \leq a$, X_2^t -terms in $(-q)^{2a+b-1}(1 - (-q)^b)D_{n,h}(a, b-1, c+2) + (-q)^{2a+2b-1}(1 - (-q)^{a-1})D_{n,h}(a-1, b, c+2)$ are

$$(10.16) \quad \begin{cases} \{\kappa_{a-1,t-1}(-q)^{(a+b-h)(t-1)+2a+b-1}(1 - (-q)^{a+b-1}) \\ -\kappa_{a-1,t-2}(-q)^{(a+b-h)(t-2)+4a+2b-h-3}(1 - (-q)^b)\}X_2^t & \text{if } t \neq a, \\ -\kappa_{a-1,a-2}(-q)^{(a+b-h)(a-2)+4a+2b-h-3}(1 - (-q)^b)X_2^a & \text{if } t = a. \end{cases}$$

First, assume that $a + b \geq h + 2$. Then, by (10.9), (10.14), (10.15), and (10.16), we have that the degree t -terms of $F_{n,h}(a, b, c)$ are as follows: for $1 \leq t \leq a-1$, we have

$$(10.17) \quad \begin{aligned} & \prod_{l=h-t+2}^{a+b-t-1}(1 - (-q)^l) \\ & \times \left\{ (-q)^{a-1} \{ \kappa_{a-1,t-1}(-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h-1} \kappa_{a-1,t-2}(-q)^{(a+b-h+1)(t-2)} \} \right. \\ & \times (1 - (-q)^{a+b-t+1})(1 - (-q)^{a+b-t}) \\ & \left. - (-q)^{a-1} \{ \kappa_{a-1,t-1}(-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h-1} \kappa_{a-1,t-2}(-q)^{(a+b-h+1)(t-2)} \} \right. \\ & \times (1 - (-q)^{a+b-t}) \\ & \left. + \{ \kappa_{a-1,t-1}(-q)^{(a+b-h)(t-1)+2a+b-1}(1 - (-q)^{a+b-1}) \right. \\ & \left. - \kappa_{a-1,t-2}(-q)^{(a+b-h)(t-2)+4a+2b-h-3}(1 - (-q)^b) \right\} \\ & = (-q)^{(a+b-h)(t-1)+3a+2b-t-3} \\ & \{ -((-q)^{t+1} - (-q)^2) \kappa_{a-1,t-1} + ((-q)^t - (-q)^a) \kappa_{a-1,t-2} \}. \end{aligned}$$

Now, by (10.11), we have that (10.17) is zero.

For $t = a$, we have that the degree a -terms of $F_{n,h}(a, b, c)$ are

$$\begin{aligned} & \prod_{l=h-a+2}^{b-1}(1 - (-q)^l) \left\{ -(-q)^{3a+b-h-2} \kappa_{a-1,a-2}(-q)^{(a+b-h+1)(a-2)}(1 - (-q)^{b+1})(1 - (-q)^b) \right. \\ & \quad \left. + (-q)^{3a+b-h-2} \kappa_{a-1,a-2}(-q)^{(a+b-h+1)(a-2)}(1 - (-q)^b) \right. \\ & \quad \left. - \kappa_{a-1,a-2}(-q)^{(a+b-h)(a-2)+4a+2b-h-3}(1 - (-q)^b) \right\} \\ & = 0. \end{aligned}$$

This shows that $F_{n,h}(a, b, c) = 0$ holds when $a + b \geq h + 2$.

Now, the remaining cases are $a + b < h - 1$, $a + b = h - 1$, h , and $h + 1$.

If $a + b < h - 1$, then X_0^t, X_1^t , and X_2^t are all zero, and hence $F_{n,h}(a, b, c) = 0$ holds.

If $a + b = h - 1$, we have that $X_0^t = 1$, and $X_1^t = X_2^t = 0$ by Theorem 9.8. Therefore, we have that $F_{n,h}(a, b, c)$ is

$$(1 - (-q)^{a+b-h+2}) \dots (1 - (-q)^{2a+b-h}) + ((-q)^{a-1} - 1)(1 - (-q)^{a+b-h+2}) \dots (1 - (-q)^{2a+b-h-1}) = 0 \quad (\text{since } 2a + b - h = a - 1).$$

This shows that $F_{n,h}(a, b, c) = 0$ holds when $a + b = h - 1$.

If $a + b = h$, then we have $X_0^t = X_1^t = 1$ and $X_2^t = 0$ by Theorem 9.8. Therefore, we have that $F_{n,h}(a, b, c)$ is

$$\begin{aligned} & (1 - (-q)^{a+b-h+2}) \dots (1 - (-q)^{2a+b-h}) + ((-q)^{a-1} - 1)(1 - (-q)^{a+b-h+2}) \dots (1 - (-q)^{2a+b-h-1}) \\ & - (-q)^{a-1}(1 - (-q)^{a+b-h+1}) \dots (1 - (-q)^{2a+b-h-1}) \\ & = 0. \end{aligned}$$

This shows that $F_{n,h}(a, b, c) = 0$ holds when $a + b = h$.

When $a + b = h + 1$, we have $X_0^t = (1 - (-q)^{h-t+2})$, and $X_1^t = X_2^t = 1$. Therefore, by (10.13), we have that $F_{n,h}(a, b, c)$ is

$$\begin{aligned} & (-q)^{a-1}(1 - (-q)^2) \dots (1 - (-q)^{a-1})(1 - (-q)^a - (-q)^{h+1} + (-q)^{h+2}) \\ & - (-q)^{a-1}(1 - (-q)^2) \dots (1 - (-q)^a) \\ & + (-q)^{2a+b-1}(1 - (-q)^b)(1 - (-q)) \dots (1 - (-q)^{a-1}) \\ & + (-q)^{2a+2b-1}(1 - (-q)^{a-1})(1 - (-q)) \dots (1 - (-q)^{a-2}) \\ & = 0. \end{aligned}$$

Therefore, $F_{n,h}(a, b, c) = 0$ holds when $a + b = h + 1$, and this finishes the proof of the lemma. \square

Lemma 10.17. *Assume that $a = 1$. Then, we have*

$$F_{n,h}(a, b, c) = \begin{cases} 0 & \text{if } b \geq h + 1 \text{ or } b \leq h - 3, \\ 1 & \text{if } b = h - 2, \\ 0 & \text{if } b = h - 1, \\ q^{2h+1} & \text{if } b = h. \end{cases}$$

Proof. Note that when $a = 1$, we have that $F_{n,h}(a, b, c)$ is

$$D_{n,h}(1, b + 1, c) - D_{n,h}(1, b, c + 1) + (-q)^{b+1}(1 - (-q)^b)D_{n,h}(1, b - 1, c + 2).$$

If $b \leq h - 3$, we have that $D_{n,h}(1, b + 1, c) = D_{n,h}(1, b, c + 1) = D_{n,h}(1, b - 1, c + 2) = 0$.

If $b \geq h + 1$, by Theorem 9.8, we have

$$\begin{aligned} & D_{n,h}(1, b + 1, c) - D_{n,h}(1, b, c + 1) + (-q)^{b+1}(1 - (-q)^b)D_{n,h}(1, b - 1, c + 2) \\ & = \prod_{l=h+1}^{b+1}(1 - (-q)^l) - \prod_{l=h+1}^b(1 - (-q)^l) + (-q)^{b+1}(1 - (-q)^b) \prod_{l=h+1}^{b-1}(1 - (-q)^l) = 0. \end{aligned}$$

If $b = h - 2$, then we have

$$D_{n,h}(1, b + 1, c) - D_{n,h}(1, b, c + 1) + (-q)^{b+1}(1 - (-q)^b)D_{n,h}(1, b - 1, c + 2) = 1 - 0 + 0 = 1.$$

If $b = h - 1$, then we have

$$D_{n,h}(1, b + 1, c) - D_{n,h}(1, b, c + 1) + (-q)^{b+1}(1 - (-q)^b)D_{n,h}(1, b - 1, c + 2) = 1 - 1 + 0 = 0.$$

If $b = h$, then we have

$$\begin{aligned} & D_{n,h}(1, b + 1, c) - D_{n,h}(1, b, c + 1) + (-q)^{b+1}(1 - (-q)^b)D_{n,h}(1, b - 1, c + 2) \\ & = (1 - (-q)^{h+1}) - 1 + (-q)^{h+1}(1 - (-q)^h) = q^{2h+1}. \end{aligned}$$

This finishes the proof of the lemma. \square

Theorem 10.18. *Assume that $x \perp L^b$, $\text{val}((x, x)) \leq -2$, and $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$. Let λ be the fundamental invariants of L^b and $(a, b, c) = (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$. Assume further that $(a, b, c) \neq (1, h, n - h - 2), (1, h - 2, n - h), (0, h + 1, n - h - 2), (0, h, n - h - 1), (0, h - 1, n - h)$. Then, we have*

$$\widehat{\partial \text{Den}}_{L^b \circ}^{n,h}(x) = 0.$$

Proof. Recall that by (10.1), we have

$$\widehat{\partial\text{Den}}_{L^{b\circ}}^{n,h}(x) = \sum_{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle).$$

Since $\text{val}(\langle x, x \rangle) \leq -2$, we have that $\text{val}(\langle u^b, u^\perp \rangle) \geq 2$. Therefore, by Proposition 10.3, $D_{n,h}(L^b + \langle u^b + u^\perp \rangle)$ depends only on u^b . Also, we have that

$$\text{vol}(L^b + \langle u^b + u^\perp \rangle) = \text{vol}(L^b) \text{vol}(\langle u^\perp \rangle).$$

This implies that

$$\begin{aligned} \widehat{\partial\text{Den}}_{L^{b\circ}}^{n,h}(x) &= \sum_{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle) \\ &= \text{vol}(L^b) \text{vol}(\langle x \rangle^\vee) \sum_{i \geq 0} q^{-2i} \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\ &= \text{vol}(L^b) \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1} \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle). \end{aligned}$$

Now, it suffices to show that $\sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = 0$.

We claim that $\sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle)$ is a weighted sum of two functions $E_{n,h}(a, b, c)$ and $F_{n,h}(a, b, c)$ as defined in Definition 10.13. More precisely, we claim that

$$(10.18) \quad \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = \mu(L_{(\lambda \geq 2) - 2}) \times E_{n,h}(a, b, c) + \{q^{2t \geq 3(\lambda)} \mu(L_{(\lambda \geq 3) - 3}) - (-q)^{t \geq 2(\lambda)} \mu(L_{(\lambda \geq 2) - 2})\} \times F_{n,h}(a, b, c).$$

Note that if the claim (10.18) holds, the theorem follows from Lemmas 10.14, 10.15, 10.16 and 10.17.

Now we prove the claim. Note that both sides of (10.18) are linear sums of the Cho-Yamauchi constants $D_{n,h}(a+1, b, c)$, $D_{n,h}(a, b+1, c)$, $D_{n,h}(a, b, c+1)$, $D_{n,h}(a+1, b-2, c+2)$, $D_{n,h}(a, b-1, c+2)$, $D_{n,h}(a-1, b+2, c)$, $D_{n,h}(a-1, b, c+2)$. Therefore, it suffices to show that both sides have the same coefficients.

First, consider the coefficients of $D_{n,h}(a+1, b, c)$ on both sides. By Proposition 10.7 (Case 1-1), we have that the coefficient of $D_{n,h}(a+1, b, c)$ on the left-hand side of (10.18) is $\mu(L_{(\lambda \geq 2) - 2})$. Also, the coefficients of $D_{n,h}(a+1, b, c)$ on the right hand side of (10.18) is $\mu(L_{\lambda \geq 2 - 2})$ (note that $D_{n,h}(a+1, b, c)$ appears only in $E_{n,h}(a, b, c)$). Therefore, the coefficients of $D_{n,h}(a+1, b, c)$ on both sides are the same.

For $D_{n,h}(a, b+1, c)$, by Proposition 10.7 (Case 1-2), we have that the coefficient of $D_{n,h}(a, b+1, c)$ on the left hand side of (10.18) is $q^{2t \geq 3(\lambda)} \mu(L_{(\lambda \geq 3) - 3}) - \mu(L_{(\lambda \geq 2) - 2})$. Now, the coefficient of $D_{n,h}(a, b+1, c)$ on the right hand side of (10.18) is

$$\begin{aligned} &\mu(L_{(\lambda \geq 2) - 2}) \{(-q)^{t \geq 2(\lambda)} - 1\} + \{q^{2t \geq 3(\lambda)} \mu(L_{(\lambda \geq 3) - 3}) - (-q)^{t \geq 2(\lambda)} \mu(L_{(\lambda \geq 2) - 2})\} \\ &= q^{2t \geq 3(\lambda)} \mu(L_{(\lambda \geq 3) - 3}) - \mu(L_{(\lambda \geq 2) - 2}). \end{aligned}$$

Therefore, the coefficients of $D_{n,h}(a, b + 1, c)$ on both sides are the same.

For $D_{n,h}(a, b, c + 1)$, by Proposition 10.7 (Case 1-3+Case 3-2), we have that the coefficient of $D_{n,h}(a, b, c + 1)$ on the left hand side of (10.18) is $q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda_{\geq 2})-2}) - \mu(L_{(\lambda_{\geq 2})-1})$. On the other hand, the coefficient of $D_{n,h}(a, b, c + 1)$ on the right hand side of (10.18) is

$$\mu(L_{(\lambda_{\geq 2})-2})(-q)^{t_{\geq 2}(\lambda)}\{(-q)^{t_{\geq 2}(\lambda)} - 1\} - (-q)^{t_{\geq 2}(\lambda)-1}\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda_{\geq 3})-3}) - (-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda_{\geq 2})-2})\}.$$

Therefore, it suffices to show that

$$\mu(L_{(\lambda_{\geq 2})-1}) + (-q)^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda_{\geq 2})-2}) = (-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda_{\geq 2})-2}) + (-q)^{t_{\geq 2}(\lambda)+2t_{\geq 3}(\lambda)-1}\mu(L_{(\lambda_{\geq 3})-3}).$$

By Lemma 10.11, we have that

$$\mu(L_{(\lambda_{\geq 2})-1}) - q^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda_{\geq 2})-2}) = -(q-1)(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1},$$

and

$$\begin{aligned} (-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda_{\geq 2})-2}) + (-q)^{t_{\geq 2}(\lambda)+2t_{\geq 3}(\lambda)-1}\mu(L_{(\lambda_{\geq 3})-3}) &= (-q)^{t_{\geq 2}(\lambda)}\{-(q-1)(-q)^{|\lambda|-t_1(\lambda)-2t_{\geq 2}(\lambda)-1}\} \\ &= -(q-1)(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1}. \end{aligned}$$

This shows that the coefficients of $D_{n,h}(a, b, c + 1)$ on both sides are the same.

For $D_{n,h}(a + 1, b - 2, c + 2)$, by Proposition 10.7 (Case 2-1+Case 4-1), we have that the coefficient of $D_{n,h}(a + 1, b - 2, c + 2)$ on the left hand side of (10.18) is $q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda_{\geq 2})-2}) \times (\mu(L_{\lambda=1}) - 1)$. On the other hand, the coefficient of $D_{n,h}(a + 1, b - 2, c + 2)$ on the right hand side of (10.18) is

$$-(-q)^{2t_{\geq 2}(\lambda)}(1 - (-q)^{t_1(\lambda)})(1 - (-q)^{t_1(\lambda)-1})\mu(L_{(\lambda_{\geq 2})-2}).$$

Now, by Lemma 10.11, we have that

$$(10.19) \quad \mu(L_{\lambda=1}) - 1 = q^{2t_1(\lambda)-1} - (-q)^{t_1(\lambda)-1}(q-1) - 1 = -(1 - (-q)^{t_1(\lambda)})(1 - (-q)^{t_1(\lambda)-1}).$$

This shows that the coefficients of $D_{n,h}(a + 1, b - 2, c + 2)$ on both sides are the same.

For $D_{n,h}(a, b - 1, c + 2)$, by Proposition 10.7 (Case 2-2+Case 4-2), we have that the coefficient of $D_{n,h}(a, b - 1, c + 2)$ on the left hand side of (10.18) is

$$q^{2t_{\geq 2}(\lambda)}\{\mu(L_{(\lambda_{\geq 2})-2} \oplus L_{\lambda=1}) - \mu(L_{(\lambda_{\geq 2})-2})\mu(L_{\lambda=1})\}.$$

On the other hand, the coefficient of $D_{n,h}(a, b - 1, c + 2)$ on the right hand side is

$$\begin{aligned} &-(-q)^{2t_{\geq 2}(\lambda)+t_1(\lambda)-1}(1 - (-q)^{t_1(\lambda)})(1 - (-q)^{t_{\geq 2}(\lambda)})\mu(L_{(\lambda_{\geq 2})-2}) \\ &+ (-q)^{2t_{\geq 2}(\lambda)+t_1(\lambda)-1}(1 - (-q)^{t_1(\lambda)})\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda_{\geq 3})-3}) - (-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda_{\geq 2})-2})\}. \end{aligned}$$

Note that by Lemma 10.11, we have that

$$\mu(L_{(\lambda_{\geq 2})-2} \oplus L_{\lambda=1}) = \mu(L_{(\lambda_{\geq 3})-2} \oplus L_{\lambda=1}) = q^{2t_{\geq 3}(\lambda)+2t_1(\lambda)-1}\mu(L_{(\lambda_{\geq 3})-3}) - (q-1)(-q)^{|\lambda|-2t_{\geq 2}(\lambda)-1}.$$

Combining these with (10.19), it suffices to show that

$$-(q-1)(-q)^{|\lambda|-1} = (-q)^{2t_{\geq 2}(\lambda)+t_1(\lambda)}\mu(L_{(\lambda_{\geq 2})-2}) + (-q)^{2t_{\geq 3}(\lambda)+2t_{\geq 2}(\lambda)+t_1(\lambda)-1}\mu(L_{(\lambda_{\geq 3})-3}).$$

Since $\mu(L_{(\lambda_{\geq 2})-2}) = \mu(L_{(\lambda_{\geq 3})-2})$, this follows from Lemma 10.11. This shows that the coefficients of $D_{n,h}(a, b - 1, c + 2)$ on both sides are the same.

For $D_{n,h}(a-1, b+2, c)$, by Proposition 10.7 (Case 3-1), we have that the coefficient of $D_{n,h}(a-1, b+2, c)$ on the left hand side of (10.18) is

$$\mu(L_{(\lambda \geq 2)-1}) - q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}).$$

On the other hand, the coefficient of $D_{n,h}(a-1, b+2, c)$ on the right hand side of (10.18) is

$$((-q)^{t_{\geq 2}(\lambda)-1} - 1) \{q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2})\}.$$

Therefore, it suffices to show that

$$\mu(L_{(\lambda \geq 2)-1}) + (-q)^{2t_{\geq 2}(\lambda)-1} \mu(L_{(\lambda \geq 2)-2}) = (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) + (-q)^{2t_{\geq 3}(\lambda)+t_{\geq 2}(\lambda)-1} \mu(L_{(\lambda \geq 3)-3}).$$

Now, by Lemma 10.11, we have that

$$\mu(L_{(\lambda \geq 2)-1}) + (-q)^{2t_{\geq 2}(\lambda)-1} \mu(L_{(\lambda \geq 2)-2}) = -(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1} (q-1),$$

and

$$(-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) + (-q)^{2t_{\geq 3}(\lambda)+t_{\geq 2}(\lambda)-1} \mu(L_{(\lambda \geq 3)-3}) = (-q)^{t_{\geq 2}(\lambda)} (-(-q)^{|\lambda|-t_1(\lambda)-2t_{\geq 2}(\lambda)-1}) (q-1).$$

This shows that the coefficients of $D_{n,h}(a-1, b+2, c)$ on both sides are the same.

For $D_{n,h}(a-1, b, c+2)$, by Proposition 10.7 (Case 5), we have that the coefficient of $D_{n,h}(a-1, b, c+2)$ on the left hand side of (10.18) is

$$\mu(L_{\lambda \geq 2} \oplus L_{\lambda=1}) - q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2} \oplus L_{\lambda=1}).$$

On the other hand, the coefficient of $D_{n,h}(a-1, b, c+2)$ on the right hand side of (10.18) is

$$(-q)^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1} (1 - (-q)^{t_{\geq 2}(\lambda)-1}) \{q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2})\}.$$

Note that by Lemma 10.11, we have

$$\mu(L_{\lambda \geq 2} \oplus L_{\lambda=1}) = q^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1} \mu(L_{(\lambda \geq 2)-1}) - (q-1)(-q)^{|\lambda|-1},$$

and

$$\mu(L_{(\lambda \geq 2)-2} \oplus L_{\lambda=1}) = \mu(L_{(\lambda \geq 3)-2} \oplus L_{\lambda=1}) = q^{2t_{\geq 3}(\lambda)+2t_1(\lambda)-1} \mu(L_{(\lambda \geq 3)-3}) - (q-1)(-q)^{|\lambda|-2t_{\geq 2}(\lambda)-1}.$$

Therefore, it suffices to show that

$$\begin{aligned} & q^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1} \mu(L_{(\lambda \geq 2)-1}) - q^{2t_{\geq 3}(\lambda)+2t_{\geq 2}(\lambda)+2t_1(\lambda)-1} \mu(L_{(\lambda \geq 3)-3}) \\ &= (-q)^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1} (1 - (-q)^{t_{\geq 2}(\lambda)-1}) \{q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2})\}. \end{aligned}$$

Note that this is equivalent to

$$\begin{aligned} \mu(L_{(\lambda \geq 2)-1}) - q^{2t_{\geq 2}(\lambda)-1} \mu(L_{(\lambda \geq 2)-2}) &= (-q)^{t_{\geq 2}(\lambda)} \{\mu(L_{(\lambda \geq 2)-2}) - q^{2t_{\geq 3}(\lambda)-1} \mu(L_{(\lambda \geq 3)-3})\} \\ &= (-q)^{t_{\geq 2}(\lambda)} \{\mu(L_{(\lambda \geq 3)-2}) - q^{2t_{\geq 3}(\lambda)-1} \mu(L_{(\lambda \geq 3)-3})\}. \end{aligned}$$

Now, by Lemma 10.11, we have that both sides are equal to $-(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1} (q-1)$. This shows that the coefficients of $D_{n,h}(a-1, b, c+2)$ on both sides are the same.

This finishes the proof of (10.18). Now, the theorem follows from Lemma 10.14, Lemma 10.16, Lemma 10.15, and Lemma 10.17. \square

Theorem 10.19. Assume that $x \perp L^b$, $\text{val}(\langle x, x \rangle) \leq -2$, and $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$. Let λ be the fundamental invariants of L^b and $(a, b, c) = (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$. Then, we have

$$\widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) = \begin{cases} q^{h-1}(q+1) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (1, h, n-h-2), \\ q^{-h}(q+1) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (1, h-2, n-h), \\ q^{-(h-1)} \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (0, h-1, n-h), \\ q^{-h} \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (0, h, n-h-1), \\ q^{-(h+1)}(q^{2h+1} + (-q)^h) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (0, h+1, n-h-2). \end{cases}$$

Proof. In the proof of Theorem 10.18, we proved that

$$\widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) = \text{vol}(L^b) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle).$$

Also, in (10.18), we showed that

$$\begin{aligned} & \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\ &= \mu(L_{(\lambda \geq 2)-2}) \times E_{n,h}(a, b, c) + \{q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2})\} \times F_{n,h}(a, b, c). \end{aligned}$$

First, assume that $(a, b, c) = (1, h, n-h-2)$, $\lambda = (\alpha, \overbrace{1, \dots, 1}^h, \overbrace{0, \dots, 0}^{n-h-2})$. In this case, we know that $E_{n,h}(a, b, c) = 0$. Also, it is easy to see that $\mu(L_\alpha) = q^{2\lceil \frac{\alpha}{2} \rceil}$, and hence

$$\begin{aligned} & q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) = q^{2t_{\geq 3}(\alpha)} \mu(L_{\alpha-3}) + q \mu(L_{\alpha-2}) \\ &= \begin{cases} 1+q & \text{if } \alpha = 2, \\ q^{2+2\lceil \frac{\alpha-3}{2} \rceil} + q^{2\lceil \frac{\alpha-2}{2} \rceil+1} = q^{\alpha-2}(q+1) & \text{if } \alpha \geq 3, \end{cases} \\ &= q^{\alpha-2}(q+1). \end{aligned}$$

Now, by Lemma 10.17, we have that

$$\sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = q^{\alpha-2}(q+1)q^{2h+1}.$$

Similarly, for $(a, b, c) = (1, h-2, n-h)$, $\lambda = (\alpha, \overbrace{1, \dots, 1}^{h-2}, \overbrace{0, \dots, 0}^{n-h})$, we have that

$$\sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = q^{\alpha-2}(q+1).$$

Therefore, we have that

$$\begin{aligned} \widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) &= \text{vol}(L^b) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\ &= \begin{cases} q^{h-1}(q+1) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (1, h, n-h-2), \\ q^{-h}(q+1) \text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a, b, c) = (1, h-2, n-h). \end{cases} \end{aligned}$$

Now, assume that $a = 0$. Then, we have that

$$\begin{aligned} & \mu(L_{(\lambda \geq 2)-2}) = 1, \\ & q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - (-q)^{t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}) = 0. \end{aligned}$$

Therefore, by Lemma 10.15, we have

$$\sum_{u^b \in (L^b)^{\vee, \geq 0} / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = \begin{cases} 1 & \text{if } b = h - 1, h, \\ q^{2h+1} + (-q)^h & \text{if } b = h + 1. \end{cases}$$

This implies that

$$\begin{aligned} \widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) &= \text{vol}(L^b) \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1} \sum_{u^b \in (L^b)^{\vee, \geq 0} / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\ &= \begin{cases} q^{-(h-1)} \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1} & \text{if } (a, b, c) = (0, h - 1, n - h), \\ q^{-h} \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1} & \text{if } (a, b, c) = (0, h, n - h - 1), \\ q^{-(h+1)} (q^{2h+1} + (-q)^h) \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1} & \text{if } (a, b, c) = (0, h + 1, n - h - 2). \end{cases} \end{aligned}$$

This finishes the proof of the theorem. \square

Theorem 10.20. *Assume that $x \perp L^b$, $\text{val}((x, x)) = -1$, and $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$. Let λ be the fundamental invariants of L^b and $(a, b, c) = (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$. Assume further that $(a, b, c) \neq (1, h, n - h - 2), (1, h - 2, n - h), (0, h + 1, n - h - 2), (0, h, n - h - 1), (0, h - 1, n - h)$. Then, we have*

$$\widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) = -\frac{1}{q^h} D_{n-1, h-1}(a, b, c).$$

Proof. Recall that

$$\widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) = \sum_{(u^b, u^\perp) \in (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle).$$

Since $\text{val}((x, x)) = -1$, we have that $\text{val}((u^\perp, u^\perp)) \geq 1$. In the proof of Theorem 10.18, we proved that

$$\sum_{(u^b, u^\perp) \in (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times, \text{val}((u^\perp, u^\perp)) \geq 2} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle) = 0.$$

Therefore, we have that

$$\widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) = \sum_{\substack{(u^b, u^\perp) \in (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times, \\ \text{val}((u^\perp, u^\perp)) = 1}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle).$$

Then by Proposition 10.4, $D_{n,h}(L^b + \langle u^b + u^\perp \rangle)$ depends only on u^b . Also, we have that

$$\text{vol}(L^b + \langle u^b + u^\perp \rangle) = \text{vol}(L^b) \text{vol}(\langle u^\perp \rangle).$$

This implies that

$$\begin{aligned} \widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) &= \sum_{\substack{(u^b, u^\perp) \in (L^b)^{\vee, \geq 0} / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times, \\ \text{val}((u^\perp, u^\perp)) = 1}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle) \\ &= \text{vol}(L^b) (q)^{-1} \sum_{u^b \in (L^b)^{\vee, \geq 0} / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle). \end{aligned}$$

Note that Proposition 10.3 and Proposition 10.4 are different only when (Case 1-1) and (Case 1-2). Also, it is easy to see that

$$\begin{aligned} & \text{(Case 1-2-1)} \\ & |\{u^b \in (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 1} - (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 2} \mid \text{val}((u^b + u^\perp, u^b + u^\perp)) = 1\}| \\ &= \frac{q-2}{q-1} \text{(Case 1-2)} = \frac{q-2}{q-1} (q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - \mu(L_{(\lambda \geq 2)-2})), \end{aligned}$$

$$\begin{aligned} & \text{(Case 1-2-2)} \\ & |\{u^b \in (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 1} - (\pi^2(L_2^b)^\vee \oplus \pi(L_1^b)^\vee)^{\geq 2} \mid \text{val}((u^b + u^\perp, u^b + u^\perp)) \geq 2\}| \\ &= \frac{1}{q-1} \text{(Case 1-2)} = \frac{1}{q-1} (q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - \mu(L_{(\lambda \geq 2)-2})). \end{aligned}$$

Now, note that if $\text{val}((u^\perp, u^\perp)) \geq 2$, then $\sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = 0$. Therefore, for

$\text{val}((u^\perp, u^\perp)) = 1$, we have

$$\begin{aligned} \widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) &= \frac{1}{q} \text{vol}(L^b) \sum_{u^b \in (L^b)^\vee, \geq 0 / L^b} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\ &= \frac{1}{q} \text{vol}(L^b) \{ (D_{n,h}(a, b+1, c) - D_{n,h}(a+1, b, c)) \mu(L_{(\lambda \geq 2)-2}) \\ &\quad + (D_{n,h}(a+1, b, c) - D_{n,h}(a, b+1, c)) \frac{1}{q-1} (q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - \mu(L_{(\lambda \geq 2)-2})) \} \\ &= \frac{1}{q} \text{vol}(L^b) (D_{n,h}(a+1, b, c) - D_{n,h}(a, b+1, c)) \left\{ \frac{q^{2t_{\geq 3}(\lambda)}}{q-1} \mu(L_{(\lambda \geq 3)-3}) - \frac{q}{q-1} \mu(L_{(\lambda \geq 2)-2}) \right\}. \end{aligned}$$

By Lemma 10.11, we have

$$\frac{q^{2t_{\geq 3}(\lambda)}}{q-1} \mu(L_{(\lambda \geq 3)-3}) - \frac{q}{q-1} \mu(L_{(\lambda \geq 2)-2}) = -(-q)^{|\lambda| - b - 2a}.$$

Also, by Theorem 9.4, we have

$$D_{n,h}(a+1, b, c) - D_{n,h}(a, b+1, c) = -(-q)^{2n-h-1-b-2c} D_{n-1, h-1}(a, b, c).$$

Finally, we have that $\text{vol}(L^b) = q^{-|\lambda|}$. Therefore, we have

$$\widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) = \frac{(-q)^{|\lambda| + 2n - 2a - 2b - 2c - h - 1}}{q^{|\lambda| + 1}} D_{n-1, h-1}(a, b, c).$$

Now, note that $a + b + c = n - 1$ and $\text{val}(L^b \oplus \langle x \rangle) = |\lambda| - 1 \equiv h + 1 \pmod{2}$. This implies that

$$\widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) = -\frac{1}{q^h} D_{n-1, h-1}(a, b, c).$$

This finishes the proof of the theorem. \square

Theorem 10.21. *Assume that $x \perp L^b$, $\text{val}((x, x)) = -1$, and $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$. Let λ be the fundamental invariants of L^b and $(a, b, c) = (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$. Then, we have*

$$\widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) = \begin{cases} q^{h-2}(q+1)(1-q^{-2})^{-1} - q^{-h} D_{n-1, h-1}(a, b, c) & \text{if } (a, b, c) = (1, h, n-h-2), \\ q^{-h-1}(q+1)(1-q^{-2})^{-1} - q^{-h} D_{n-1, h-1}(a, b, c) & \text{if } (a, b, c) = (1, h-2, n-h), \\ q^{-h-1}(1-q^{-2})^{-1} - q^{-h} D_{n-1, h-1}(a, b, c) & \text{if } (a, b, c) = (0, h, n-h-1). \end{cases}$$

Note that two cases $(a, b, c) = (0, h-1, n-h), (0, h+1, n-h-2)$ are not possible since $\text{val}((x, x)) = -1$ and hence $\text{val}(L^b) \equiv h \pmod{2}$.

Proof. Recall that

$$\begin{aligned}
(10.20) \quad \widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) &= \sum_{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times ((x)^\vee \setminus \{0\}) / O_F^\times} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(L^b + \langle u^b + u^\perp \rangle) \\
&= \text{vol}(L^b) \sum_{\substack{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times ((x)^\vee \setminus \{0\}) / O_F^\times \\ \text{val}((u^\perp, u^\perp)=1}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(\langle u^\perp \rangle) \\
&+ \text{vol}(L^b) \sum_{\substack{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times ((x)^\vee \setminus \{0\}) / O_F^\times \\ \text{val}((u^\perp, u^\perp) \geq 3}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(\langle u^\perp \rangle).
\end{aligned}$$

In the proof of the Theorem 10.19, we showed that

$$\begin{aligned}
&\text{vol}(L^b) \sum_{\substack{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times ((x)^\vee \setminus \{0\}) / O_F^\times \\ \text{val}((u^\perp, u^\perp) \geq 3}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(\langle u^\perp \rangle) \\
&= \text{vol}(L^b) \frac{1}{q^3} (1 - q^{-2})^{-1} \sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}((u^\perp, u^\perp) \geq 3}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle).
\end{aligned}$$

Also, we already computed the sum $\sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}((u^\perp, u^\perp) \geq 3}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle)$ in the proof of the Theorem 10.19 (more precisely, we computed this for $\text{val}((u^\perp, u^\perp)) \geq 2$). Therefore, we only need to compute

$$\text{vol}(L^b) \sum_{\substack{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times ((x)^\vee \setminus \{0\}) / O_F^\times \\ \text{val}((u^\perp, u^\perp)=1}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(\langle u^\perp \rangle).$$

Note that Proposition 10.3 and Proposition 10.4 are different only when (Case 1-1) and (Case 1-2). Therefore, as in the proof of Theorem 10.20, we have that

$$\begin{aligned}
&\sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}((u^\perp, u^\perp)=1}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) - \sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}((u^\perp, u^\perp) \geq 2}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\
&= (D_{n,h}(a+1, b, c) - D_{n,h}(a, b+1, c)) \left\{ \frac{q^{2t_{\geq 3}(\lambda)}}{q-1} \mu(L_{(\lambda \geq 3)-3}) - \frac{q}{q-1} \mu(L_{(\lambda \geq 2)-2}) \right\} \\
&= (-(-q)^{2n-h-1-b-2c} D_{n-1, h-1}(a, b, c)) (-(-q)^{|\lambda|-b-2a}) \\
&= (-q)^{|\lambda|-h+1} D_{n-1, h-1}(a, b, c).
\end{aligned}$$

Since $\text{vol}(L^b) = q^{-|\lambda|}$ and $\text{val}(L^b \oplus \langle x \rangle) = |\lambda| - 1 \equiv h + 1 \pmod{2}$, we have that

$$\begin{aligned}
& \text{vol}(L^b) \sum_{\substack{(u^b, u^\perp) \in (L^b)^\vee, \geq 0 / L^b \times (\langle x \rangle^\vee \setminus \{0\}) / O_F^\times \\ \text{val}(\langle u^\perp, u^\perp \rangle) = 1}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \text{vol}(\langle u^\perp \rangle) \\
&= \frac{(-q)^{|\lambda| - h + 1}}{q^{|\lambda| + 1}} D_{n-1, h-1}(a, b, c) + \frac{\text{vol}(L^b)}{q} \sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}(\langle u^\perp, u^\perp \rangle) \geq 2}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) \\
&= -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) + \frac{\text{vol}(L^b)}{q} \sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}(\langle u^\perp, u^\perp \rangle) \geq 2}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle).
\end{aligned}$$

Now, the theorem follows from (10.20) and the fact that (see the proof of Theorem 10.19)

$$\sum_{\substack{u^b \in (L^b)^\vee, \geq 0 / L^b \\ \text{val}(\langle u^\perp, u^\perp \rangle) \geq 2}} D_{n,h}(L^b + \langle u^b + u^\perp \rangle) = \begin{cases} q^{h-1}(q+1)q^{|\lambda|} & \text{if } (a, b, c) = (1, h, n-h-2), \\ q^{-h}(q+1)q^{|\lambda|} & \text{if } (a, b, c) = (1, h-2, n-h-1), \\ 1 & \text{if } (a, b, c) = (0, h, n-h). \end{cases}$$

□

We will use the following lemma in the next section when we count the number of horizontal components. Recall that for $\lambda \in \mathcal{R}_n^{0+}$, we define A_λ by $\text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n})$.

Lemma 10.22. *Assume that $\lambda \geq 2$. Consider $(\lambda, 0^{n-1}), (\lambda, 2^{n-1}) \in \mathcal{R}_n^{0+}$ and $(0^{n-1}), (2^{n-1}) \in \mathcal{R}_{n-1}^{0+}$. Then, we have*

$$\frac{\text{Den}(A_{(\lambda, 0^{n-1})}, A_{(\lambda, 2^{n-1})}) / \text{Den}(A_{(\lambda, 0^{n-1})}, A_{(\lambda, 0^{n-1})})}{\text{Den}(A_{(0^{n-1})}, A_{(2^{n-1})}) / \text{Den}(A_{(0^{n-1})}, A_{(0^{n-1})})} = \begin{cases} q^{2n-2} & \text{if } \lambda \geq 3, \\ q^{2n-2} \frac{1 - (-q)^{-n}}{1 - (-q)^{-1}} & \text{if } \lambda = 2. \end{cases}$$

Proof. We use [Cho22a, Theorem 2.5] to prove this. To simplify notation, we use the following convention: For $1 \leq k \leq n$, $B_1 \in X_n(O_F)$, and $B_2 \in X_{n-1}(O_F)$, we define

$$\begin{aligned}
R_0^k &= \text{Den}(A_{(\lambda, 1^{k-1}, 0^{n-k})}, B_1), & R_1^k &= \text{Den}(A_{(\lambda, 2^{k-1}, 1^{n-k})}, B_1), & R_2^k &= \text{Den}(A_{(\lambda, 3^{k-1}, 2^{n-k})}, B_1), \\
R_3^k &= \text{Den}(A_{(1^{k-1}, 0^{n-k})}, B_2), & R_4^k &= \text{Den}(A_{(2^{k-1}, 1^{n-k})}, B_2), & R_5^k &= \text{Den}(A_{(3^{k-1}, 2^{n-k})}, B_2),
\end{aligned}$$

and for a polynomial $f(X) = \sum_{i=1}^n a_i X^i$, we denote by $f(R_i)$ the sum $f(R_i) := \sum_{i=1}^n a_i R_i^k$. For example, if $f(X) = X + X^2$, then $f(R_1) = \text{Den}(A_{(\lambda, 0^{n-1})}, B_1) + \text{Den}(A_{(\lambda, 1^1, 0^{n-2})}, B_1)$ (not $\text{Den}(A_{(\lambda, 0^{n-1})}, B_1) + \{\text{Den}(A_{(\lambda, 0^{n-1})}, B_1)\}^2$).

Also, we define the following polynomials

$$\begin{aligned}
f_{1,k}(X) &= X^k \prod_{l=k+1}^n (1 - (-q)^{-l} X), & f_{2,k}(X) &= X \prod_{l=k+1}^n (1 - (-q)^{-l} X), \\
f_{3,k}(X) &= X^k \prod_{l=k}^{n-1} (1 - (-q)^{-l} X), & f_{4,k}(X) &= X \prod_{l=k}^{n-1} (1 - (-q)^{-l} X).
\end{aligned}$$

Finally, for $0 \leq j \leq i$, we define the following constants $k_{i,j}$:

$$\prod_{l=1}^i (1 + (-q)^{-l} X) = \sum_{j=0}^i k_{i,j} X^j.$$

First, we claim that

$$(10.21) \quad f_{2,i+1}(X) = \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X), 0 \leq i \leq n-1,$$

and

$$(10.22) \quad f_{4,i+1}(X) = \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j-1)}{2}} f_{3,j+1}(X), 0 \leq i \leq n-1.$$

Let us prove (10.21) by induction on i . For $i = 0$, the claim holds since $f_{2,1}(X) = f_{1,1}(X)$. Now, assume that the claim holds for i , i.e.,

$$(10.23) \quad X \prod_{l=i+2}^n (1 - (-q)^{-l} X) = \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}} X^{j+1} \prod_{l=j+2}^n (1 - (-q)^{-l} X).$$

Note that $\sum_{j=0}^{i+1} k_{i+1,j} X^j = \prod_{l=1}^{i+1} (1 + (-q)^{-l} X) = (1 + (-q)^{-i-1} X) (\sum_{j=0}^i k_{i,j} X^j)$, and hence

$$(10.24) \quad \begin{aligned} \sum_{j=0}^{i+1} k_{i+1,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X) &= \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X) \\ &+ \sum_{j=0}^i (-q)^{-i-1} k_{i,j}(-q)^{-\frac{(j+1)(j+2)}{2}} f_{1,j+2}(X). \end{aligned}$$

Now, consider the equation (10.23) with $X \Rightarrow (-q)^{-1} X$, $n \Rightarrow n-1$. Then, we have

$$\begin{aligned} (-q)^{-1} X \prod_{l=i+3}^n (1 - (-q)^{-l} X) &= \sum_{j=0}^i k_{i,j}(-q)^{-\frac{(j+1)(j+2)}{2}} X^{j+1} \prod_{l=j+3}^n (1 - (-q)^{-l} X) \\ \iff \sum_{j=0}^i (-q)^{-i-1} k_{i,j}(-q)^{-\frac{(j+1)(j+2)}{2}} f_{1,j+2}(X) &= (-q)^{-i-2} X^2 \prod_{l=i+3}^n (1 - (-q)^{-l} X). \end{aligned}$$

Combining this with (10.24), we have

$$\begin{aligned} \sum_{j=0}^{i+1} k_{i+1,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X) &= X \prod_{l=i+2}^n (1 - (-q)^{-l} X) + (-q)^{-i-2} X^2 \prod_{l=i+3}^n (1 - (-q)^{-l} X) \\ &= X \prod_{l=i+3}^n (1 - (-q)^{-l} X). \end{aligned}$$

This finishes the proof of (10.21). The equation (10.22) can be proved in a similar way.

Now, note that [Cho22a, Theorem 2.5] implies that if $B_1 = A_{\eta_1}$, $B_2 = A_{\eta_2}$ such that $\eta_1 \geq \underbrace{(2, \dots, 2)}_n$, $\eta_2 \geq \underbrace{(2, \dots, 2)}_{n-1}$, then we have

$$(10.25) \quad \begin{aligned} f_{1,j}(R_0) &= \frac{(-1)^{n-j}}{(-q)^{n(n-j)}} f_{2,j}(R_1), & f_{1,j}(R_1) &= \frac{(-1)^{n-j}}{(-q)^{n(n-j)}} f_{2,j}(R_2), \\ f_{3,j}(R_3) &= \frac{(-1)^{n-j}}{(-q)^{(n-1)(n-j)}} f_{4,j}(R_4), & f_{3,j}(R_4) &= \frac{(-1)^{n-j}}{(-q)^{(n-1)(n-j)}} f_{4,j}(R_5). \end{aligned}$$

Furthermore, if $B_1 = A_{(\lambda, 2^{n-1})}$ and $B_2 = A_{(2^{n-1})}$, then $R_2^k = R_5^k = 0$ for all $k > 1$. Therefore,

$$(10.26) \quad f_{2,j}(R_2) = R_2 = \text{Den}(A_{(\lambda, 2^{n-1})}, A_{(\lambda, 2^{n-1})}), \quad f_{4,j}(R_5) = R_5 = \text{Den}(A_{(2^{n-1})}, A_{(2^{n-1})}).$$

Since $\text{Den}(A_{(\lambda,1,0^{n-2})}, A_{(\lambda,2^{n-1})}) = 0$, we have

$$\begin{aligned}
(10.27) \quad & \text{Den}(A_{(\lambda,0^{n-1})}, A_{(\lambda,2^{n-1})}) = R_0(1 - (-q)^{-n}R_0) = f_{2,n-1}(R_0) \\
& = \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} f_{1,i+1}(R_0) && \text{(by (10.21))} \\
& = \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{n(n-i-1)}} f_{2,i+1}(R_1) && \text{(by (10.25))} \\
& = \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{n(n-i-1)}} \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(R_1) && \text{(by (10.21))} \\
& = \left\{ \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{n(n-i-1)}} \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}} \frac{(-1)^{n-j-1}}{(-q)^{n(n-j-1)}} \right\} R_2 && \text{(by (10.25), (10.26)).}
\end{aligned}$$

Similarly, note that $\text{Den}(A_{(1,0^{n-2})}, A_{(2^{n-1})}) = 0$, and hence

$$\begin{aligned}
(10.28) \quad & \text{Den}(A_{(0^{n-1})}, A_{(2^{n-1})}) = R_3(1 - (-q)^{-(n-1)}R_3) = f_{4,n-1}(R_3) \\
& = \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} f_{3,i+1}(R_3) && \text{(by (10.22))} \\
& = \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{(n-1)(n-i-1)}} f_{4,i+1}(R_4) && \text{(by (10.25))} \\
& = \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{(n-1)(n-i-1)}} \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j-1)}{2}} f_{3,j+1}(R_4) && \text{(by (10.22))} \\
& = \left\{ \sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{(n-1)(n-i-1)}} \sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j-1)}{2}} \frac{(-1)^{n-j-1}}{(-q)^{(n-1)(n-j-1)}} \right\} R_5 && \text{(by (10.25), (10.26)).}
\end{aligned}$$

Comparing (10.27) and (10.28), we have

$$(10.29) \quad (-q)^{2n-2} \frac{\text{Den}(A_{(\lambda,0^{n-1})}, A_{(\lambda,2^{n-1})})}{\text{Den}(A_{(\lambda,2^{n-1})}, A_{(\lambda,2^{n-1})})} = \frac{\text{Den}(A_{(0^{n-1})}, A_{(2^{n-1})})}{\text{Den}(A_{(2^{n-1})}, A_{(2^{n-1})})}.$$

Now, by [Cho23, Lemma 4.6], we have

$$\begin{aligned}
& \text{Den}(A_{(2^{n-1})}, A_{(2^{n-1})}) = q^{2(n-1)^2} \text{Den}(A_{(0^{n-1})}, A_{(0^{n-1})}) = q^{2(n-1)^2} \prod_{l=1}^{n-1} (1 - (-q)^{-l}), \\
& \text{Den}(A_{(\lambda,(0^{n-1})}, A_{(\lambda,0^{n-1})})} = q^\lambda (1 - (-q)^{-1}) \prod_{l=1}^{n-1} (1 - (-q)^{-l}), \text{ if } \lambda \geq 1, \\
& \text{Den}(A_{(\lambda,2^{n-1})}, A_{(\lambda,2^{n-1})}) = \begin{cases} q^{2(n^2-1)+\lambda} (1 - (-q)^{-1}) \prod_{l=1}^{n-1} (1 - (-q)^{-l}) & \text{if } \lambda \geq 3, \\ q^{2n^2} \prod_{l=1}^n (1 - (-q)^{-l}) & \text{if } \lambda = 2. \end{cases}
\end{aligned}$$

Combining these with (10.29), we have

$$\begin{aligned}
& \frac{\text{Den}(A_{(\lambda,0^{n-1})}, A_{(\lambda,2^{n-1})}) / \text{Den}(A_{(\lambda,0^{n-1})}, A_{(\lambda,0^{n-1})})}{\text{Den}(A_{(0^{n-1})}, A_{(2^{n-1})}) / \text{Den}(A_{(0^{n-1})}, A_{(0^{n-1})})} \\
& = (-q)^{-2n+2} \frac{\text{Den}(A_{(\lambda,2^{n-1})}, A_{(\lambda,2^{n-1})}) \text{Den}(A_{(0^{n-1})}, A_{(0^{n-1})})}{\text{Den}(A_{(\lambda,0^{n-1})}, A_{(\lambda,0^{n-1})}) \text{Den}(A_{(2^{n-1})}, A_{(2^{n-1})})} = \begin{cases} q^{2n-2} & \text{if } \lambda \geq 3, \\ q^{2n-2} \frac{1 - (-q)^{-n}}{1 - (-q)^{-1}} & \text{if } \lambda = 2. \end{cases}
\end{aligned}$$

This finishes the proof of the lemma. \square

11. TATE CONJECTURES AND THE PROOF OF THE MAIN THEOREM

11.1. The proof of the main theorem. Consider the Rapoport–Zink space $\mathcal{N}_n^{[h]}$ and the space of special homomorphisms \mathbb{V} . Assume that $L^b \subset \mathbb{V}$ is an O_F -lattice of rank $n - 1$. Recall that for L^b such that $L^b \subset L^b \subset (L^b)^\vee \subset L_F^b$, we define the primitive part $\mathcal{Z}(L^b)^\circ$ of $\mathcal{Z}(L^b)$ inductively by setting

$$\mathcal{Z}(L^b)^\circ := \mathcal{Z}(L^b) - \sum_{\substack{L'^b \subset L^b \\ L'^b \subset (L'^b)^\vee \subset L_F^b}} \mathcal{Z}(L'^b)^\circ.$$

For example, for a lattice L^b with fundamental invariants $(0, 0, \dots, 0, 1)$, there is no integral lattice L'^b such that $L^b \subset L'^b$ and hence $\mathcal{Z}(L^b)^\circ = \mathcal{Z}(L^b)$. For a lattice L^b with fundamental invariants $(0, 0, \dots, 0, 3)$, there is one integral lattice L'^b satisfying the above conditions, and its fundamental invariants are $(0, 0, \dots, 0, 1)$. Therefore, we have $\mathcal{Z}(L^b)^\circ = \mathcal{Z}(L^b) - \mathcal{Z}(L'^b)^\circ = \mathcal{Z}(L^b) - \mathcal{Z}(L'^b)$.

Now, let us define the derived special cycles $\mathbb{L}\mathcal{Z}(L)$.

Definition 11.1. For a lattice $L \subset \mathbb{V}$, choose its basis x_1, \dots, x_r . We define the derived special cycle $\mathbb{L}\mathcal{Z}(L)$ as the image of $O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_r)}$ in the r -th graded piece of the Grothendieck group $\mathrm{Gr}^r K_0^{\mathcal{Z}(L)}(\mathcal{N})$. This does not depend on the choice of the basis by Proposition 2.12.

By an induction on $\mathrm{val}(L)$, Conjecture 7.7 is equivalent to the following statement: for $x \in \mathbb{V} \setminus L_F^b$,

$$(11.1) \quad \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) = \partial \mathrm{Den}_{L^b}^{n,h}(x) := \sum_{L^b \subset L' \subset L'^b, L' \cap L_F^b = L^b} D_{n,h}(L') 1_{L'}(x),$$

where $L' \subset \mathbb{V}$ are O_F -lattices of rank n . For example, for a lattice L^b with fundamental invariants $(0, \dots, 0, 1)$, it is obvious that Conjecture 7.7 is equivalent to (11.1). For a lattice L^b with fundamental invariants $(0, \dots, 0, 3)$, there is one integral lattice L'^b such that $L^b \subsetneq L'^b$, and the fundamental invariants of L'^b is $(0, \dots, 0, 1)$. Therefore, Conjecture 7.7 is equivalent to

$$\begin{aligned} \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) &= \sum_{L^b \subset L' \subset L'^b} D_{n,h}(L') 1_{L'}(x) - \sum_{L^b \subset L' \subset L'^b, L' \cap L_F^b = L'^b} D_{n,h}(L') 1_{L'}(x) \\ &= \sum_{L^b \subset L' \subset L'^b, L' \cap L_F^b = L^b} D_{n,h}(L') 1_{L'}(x), \end{aligned}$$

which is equivalent to (11.1).

Now, note that there is a decomposition of the derived special cycle $\mathbb{L}\mathcal{Z}(L^b)$ into a sum of horizontal and vertical parts (see Section 2.3):

$$\mathbb{L}\mathcal{Z}(L^b) = \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}} + \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{V}}.$$

We denote by $\mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}^\circ$ (resp. $\mathbb{L}\mathcal{Z}(L^b)_{\mathcal{V}}^\circ$) the primitive part of $\mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}$ (resp. $\mathbb{L}\mathcal{Z}(L^b)_{\mathcal{V}}$).

We define

$$\begin{aligned} \mathrm{Int}_{L^b, \mathcal{H}}(x) &:= \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}), & \mathrm{Int}_{L^b, \mathcal{V}}(x) &:= \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{V}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}), \\ \mathrm{Int}_{L^b, \mathcal{H}}^\circ(x) &:= \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}), & \mathrm{Int}_{L^b, \mathcal{V}}^\circ(x) &:= \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{V}}^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}). \end{aligned}$$

Then, Conjecture 7.6 is equivalent to

$$(11.2) \quad \mathrm{Int}_{L^b, \mathcal{H}}(x) + \mathrm{Int}_{L^b, \mathcal{V}}(x) = \partial \mathrm{Den}_{L^b}^{n,h}(x).$$

or

$$(11.3) \quad \mathrm{Int}_{L^b, \mathcal{H}}^\circ(x) + \mathrm{Int}_{L^b, \mathcal{V}}^\circ(x) = \partial \mathrm{Den}_{L^b}^{n,h}(x).$$

Now, we define $\partial \mathrm{Den}_{L^b, \mathcal{V}}^{n,h}(x)$ by

$$\partial \mathrm{Den}_{L^b, \mathcal{V}}^{n,h}(x) := \partial \mathrm{Den}_{L^b}^{n,h}(x) - \mathrm{Int}_{L^b, \mathcal{H}}^\circ(x).$$

The reason we define the vertical part in this way is that it cannot be separated solely by the invariants of L^b . For example, even for a certain lattice $L^b \in H(\mathbb{V})$, $\mathbb{L}\mathcal{Z}(L^b)^\circ$ consists of both

vertical and horizontal parts. Therefore, the corresponding analytic part should include both parts as well.

Let $\lambda \in \mathcal{R}_{n-1}^{0+}$ be the fundamental invariants of L^b . Then, by Lemma 5.6, we know that if $L^b \notin H(\mathbb{V})$ (i.e., $(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)) \neq (1, h, n-h-2), (1, h-2, n-h), (0, h-1, n-h), (0, h, n-h-1), (0, h+1, n-h-2))$), then we have

$$\text{Int}_{L^{b\circ}, \mathcal{H}}(x) = 0,$$

and hence we have that

$$\partial \text{Den}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \partial \text{Den}_{L^{b\circ}}^{n,h}(x),$$

in this case.

When $L^b \in H(\mathbb{V})$, then the horizontal part of $\mathcal{Z}(L^b)$ is not empty, and hence we need to be careful. Note that by Theorem 5.3, the horizontal part of $\mathcal{Z}(L^b)^\circ$ comes from $\mathcal{N}_2^{[0]}$ or $\mathcal{N}_2^{[2]}$. By [KR11, Theorem 1.1] and reductions in Proposition 2.7, we have a precise formula for $\text{Int}_{L^{b\circ}, \mathcal{H}}(x)$.

Now, we will prove the following theorem.

Theorem 11.2. (cf. [LZ22a, Theorem 7.4.1]) *Assume that $x \perp L^b$ and $(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)) = (a, b, c)$ where $\lambda \in \mathcal{R}_{n-1}^{0+}$ is the fundamental invariants of L^b . Then, we have the followings.*

- (1) *If $\text{val}((x, x)) \leq -2$, we have $\widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = 0$.*
- (2) *If $\text{val}((x, x)) = -1$, we have*

$$\widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \begin{cases} -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) & \text{if } (a, b, c) \neq (1, h-2, n-h), \\ -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) + \frac{1}{q^h} & \text{if } (a, b, c) = (1, h-2, n-h). \end{cases}$$

Proof. First, if $(a, b, c) := (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)) \neq (1, h, n-h-2), (1, h-2, n-h), (0, h-1, n-h), (0, h, n-h-1), (0, h+1, n-h-2)$, then the assertion (1) follows from Theorem 10.18, and the assertion (2) follows from Theorem 10.20.

Now, assume that $(a, b, c) = (1, h, n-h-2), (1, h-2, n-h), (0, h-1, n-h), (0, h, n-h-1), (0, h+1, n-h-2)$. Since we already know the Fourier transform of $\partial \text{Den}_{L^{b\circ}}^{n,h}(x)$ by Theorem 10.19 and Theorem 10.21, we only need to know the Fourier transform of $\text{Int}_{L^{b\circ}, \mathcal{H}}(x)$.

When $(a, b, c) = (1, h, n-h-2)$, let us write $L^b = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are π^λ ($\lambda \geq 2$), $\pi I_h, I_{n-h-2}$, respectively. Then, by Proposition 2.7, $\mathcal{Z}(L^b)^\circ_{\mathcal{H}}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathcal{H}}$ in $N_{h+2}^{[h]}$. Therefore, by Theorem 5.3, we have

$$(11.4) \quad \mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathcal{H}} = \sum_{\substack{L_2 \oplus L_1 \subset L_2 \oplus N \subset \pi^{-1}(L_2 \oplus L_1) \\ N \simeq (\pi^{-1})^h}} \mathcal{Z}(L_2)^\circ \cdot \mathcal{Y}(N)^\circ.$$

By Proposition 2.7 again, $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathcal{H}}$ can be reduced to $\mathcal{Z}(L_2)^\circ_{\mathcal{H}} = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[0]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[0]}$, and hence

$$\text{Int}_{L_2, \mathcal{H}}(x) = \text{Int}_{L_2}(x) = \partial \text{Den}_{L_2}^{2,0}(x).$$

By Theorem 10.19 and Theorem 10.21, we have that

$$\widehat{\text{Int}}_{L_2^\circ}(x) = \widehat{\partial\text{Den}}_{L_2^\circ}^{2,0}(x) = q^{-1}(q+1)\text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(N_2)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^h} \widehat{\text{Int}}_{L_2^\circ}(x) = q^{-h-1}(q+1)\text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by Lemmas 3.7 and 10.22, the number of lattices $L_2 \oplus N$ in the sum in (11.4) is

$$\begin{aligned} & \frac{\text{Den}(A_{(\lambda,(-1)^h)}, A_{(\lambda,1^h)})/\text{Den}(A_{(\lambda,(-1)^h}), A_{(\lambda,(-1)^h)})}{\text{Den}(A_{((-1)^h)}, A_{(1^h)})/\text{Den}(A_{((-1)^h)}, A_{((-1)^h)})} \\ &= \frac{\text{Den}(A_{((\lambda+1),0^h)}, A_{((\lambda+1),2^h)})/\text{Den}(A_{((\lambda+1),0^h)}, A_{((\lambda+1),0^h)})}{\text{Den}(A_{(0^h)}, A_{(2^h)})/\text{Den}(A_{(0^h)}, A_{(0^h)})} = q^{2h}. \end{aligned}$$

This implies that

$$\begin{aligned} \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) &= q^{2h} q^{-h-1}(q+1)\text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} \\ &= q^{h-1}(q+1)\text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1}. \end{aligned}$$

Now, by Theorem 10.19, Theorem 10.21, we have that

$$\widehat{\partial\text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \widehat{\partial\text{Den}}_{L^{b\circ}}^{n,h}(x) - \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) = \begin{cases} 0 & \text{if } \text{val}(\langle x, x \rangle) \leq -2, \\ -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) & \text{if } \text{val}(\langle x, x \rangle) = -1. \end{cases}$$

This proves (1) and (2) when $(a, b, c) = (1, h, n-h-2)$.

Assume that $(a, b, c) = (1, h-2, n-h)$ and let us write $L^b = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are π^λ ($\lambda \geq 2$), πI_{h-2} , I_{n-h} , respectively. Then, by Proposition 2.7, $\mathcal{Z}(L^b)_{\mathcal{H}}^\circ$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^\circ$ in $\mathcal{N}_h^{[h]}$. Also, in $\mathcal{N}_h^{[h]}$, we have that $\mathcal{Z}(w) = \mathcal{Y}(\pi^{-1}w)$ for any $w \in \mathbb{V}$, and hence $\mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^\circ$ can be reduced to $\mathcal{Z}(L_2)_{\mathcal{H}}^\circ = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[2]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[2]}$, and hence

$$\text{Int}_{L_2^\circ, \mathcal{H}}(x) = \text{Int}_{L_2^\circ}(x) = \partial\text{Den}_{L_2^\circ}^{2,2}(x).$$

By Theorem 10.19 and Theorem 10.21, we have that

$$\widehat{\text{Int}}_{L_2^\circ}(x) = \widehat{\partial\text{Den}}_{L_2^\circ}^{2,2}(x) = \begin{cases} q^{-2}(q+1)\text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } \text{val}(\langle x, x \rangle) \leq -2, \\ q^{-3}(q+1)(1-q^{-2})^{-1} - q^{-2} & \text{if } \text{val}(\langle x, x \rangle) = -1. \end{cases}$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[2]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(\pi^{-1}L_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^{h-2}} \widehat{\text{Int}}_{L_2^\circ}(x) = \begin{cases} q^{-h}(q+1)\text{vol}(\langle x \rangle^\vee)(1-q^{-2})^{-1} & \text{if } \text{val}(\langle x, x \rangle) \leq -2, \\ q^{-h-1}(q+1)(1-q^{-2})^{-1} - q^{-h} & \text{if } \text{val}(\langle x, x \rangle) = -1. \end{cases}$$

Now, by Theorem 10.19, Theorem 10.21, we have that

$$\widehat{\partial\text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \widehat{\partial\text{Den}}_{L^{b\circ}}^{n,h}(x) - \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) = \begin{cases} 0 & \text{if } \text{val}(\langle x, x \rangle) \leq -2, \\ -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) + \frac{1}{q^h} & \text{if } \text{val}(\langle x, x \rangle) = -1. \end{cases}$$

This proves (1) and (2) when $(a, b, c) = (1, h-2, n-h)$.

Assume that $(a, b, c) = (0, h - 1, n - h)$ and let us write $L^b = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are $\pi I_1, \pi I_{h-2}, I_{n-h}$, respectively. Note that in this case, $\text{val}(\langle x, x \rangle) \neq -1$ since $\text{val}(L^b \oplus \langle x \rangle) \equiv h + 1 \pmod{2}$. By Proposition 2.7, $\mathcal{Z}(L^b)_{\mathcal{H}}^\circ$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^\circ$ in $\mathcal{N}_h^{[h]}$. Also, in $\mathcal{N}_h^{[h]}$, we have that $\mathcal{Z}(w) = \mathcal{Y}(\pi^{-1}w)$ for any $w \in \mathbb{V}$, and hence $\mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^\circ$ can be reduced to $\mathcal{Z}(L_2)_{\mathcal{H}}^\circ = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[2]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[2]}$, and hence

$$\text{Int}_{L_2, \mathcal{H}}(x) = \text{Int}_{L_2^\circ}(x) = \partial \text{Den}_{L_2^\circ}^{2,2}(x).$$

By Theorem 10.19, we have that

$$\widehat{\text{Int}}_{L_2^\circ}(x) = \widehat{\partial \text{Den}}_{L_2^\circ}^{2,2}(x) = q^{-1} \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[2]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(\pi^{-1}L_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^{h-2}} \widehat{\text{Int}}_{L_2^\circ}(x) = q^{-h+1} \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1}.$$

By Theorem 10.19 we have that

$$\widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) - \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) = 0,$$

This proves (1) when $(a, b, c) = (0, h - 1, n - h)$.

Assume that $(a, b, c) = (0, h, n - h - 1)$ and let us write $L^b = L_1 \oplus L_0$, where the hermitian matrix of L_1, L_0 are $\pi I_h, I_{n-h-1}$, respectively. By Proposition 2.7, $\mathcal{Z}(L^b)_{\mathcal{H}}^\circ$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_1)_{\mathcal{H}}^\circ$ in $\mathcal{N}_{h+1}^{[h]}$. Therefore, by Theorem 5.3, we have that $\mathcal{Z}(L_1)_{\mathcal{H}}^\circ = \mathcal{Y}(\pi^{-1}L_1)$ and hence $\langle \mathcal{Y}(\pi^{-1}L_1), \mathcal{Z}(x) \rangle$ is $\langle \mathcal{Z}(x) \rangle$ in $\mathcal{N}_1^{[0]}$. Now, By [KR11, Theorem 1.1] and Proposition 2.7, we know that $\langle \mathcal{Z}(x) \rangle = \partial \text{Den}_\emptyset^{1,0}(x)$. By Theorem 10.19 and Theorem 10.21, we have that

$$\widehat{\partial \text{Den}}_\emptyset^{1,0}(x) = \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_1^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Y}(\pi^{-1}L_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^h} \widehat{\partial \text{Den}}_\emptyset^{1,0}(x) = q^{-h} \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1}.$$

This implies that

$$\widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) = q^{-h} \text{vol}(\langle x \rangle^\vee) (1 - q^{-2})^{-1}$$

Now, by Theorem 10.19, Theorem 10.21, we have that

$$\widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) - \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) = \begin{cases} 0 & \text{if } \text{val}(\langle x, x \rangle) \leq -2, \\ -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) & \text{if } \text{val}(\langle x, x \rangle) = -1. \end{cases}$$

This proves (1) and (2) when $(a, b, c) = (0, h, n - h - 1)$.

When $(a, b, c) = (0, h + 1, n - h - 2)$, let us write $L^b = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are $\pi^1 I_1, \pi I_h, I_{n-h-2}$, respectively. Note that in this case, $\text{val}(\langle x, x \rangle) \neq -1$ since

$\text{val}(L^b \oplus \langle x \rangle) \equiv h+1 \pmod{2}$. By Proposition 2.7, $\mathcal{Z}(L^b)_{\mathcal{H}}^{\circ}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^{\circ}$ in $\mathcal{N}_{h+2}^{[h]}$. Therefore, by Theorem 5.3, we have

$$(11.5) \quad \mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^{\circ} = \sum_{\substack{L_2 \oplus L_1 \subset L_2 \oplus N \subset \pi^{-1}(L_2 \oplus L_1) \\ N \simeq (\pi^{-1})^h}} \mathcal{Z}(L_2)^{\circ} \cdot \mathcal{Y}(N)^{\circ}.$$

By Proposition 2.7 again, $\mathcal{Z}(L_2 \oplus L_1)_{\mathcal{H}}^{\circ}$ can be reduced to $\mathcal{Z}(L_2)_{\mathcal{H}}^{\circ} = \mathcal{Z}(L_2)^{\circ}$ in $\mathcal{N}_2^{[0]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[0]}$, and hence

$$\text{Int}_{L_2, \mathcal{H}}(x) = \text{Int}_{L_2^{\circ}}(x) = \partial \text{Den}_{L_2^{\circ}}^{2,0}(x).$$

By Theorem 10.19 and Theorem 10.21, we have that

$$\widehat{\text{Int}}_{L_2^{\circ}}(x) = \widehat{\partial \text{Den}}_{L_2^{\circ}}^{2,0}(x) = q^{-1}(q+1) \text{vol}(\langle x \rangle^{\vee})(1-q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^{\circ}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(N_2)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^h} \widehat{\text{Int}}_{L_2^{\circ}}(x) = q^{-h-1}(q+1) \text{vol}(\langle x \rangle^{\vee})(1-q^{-2})^{-1}.$$

Now, by Lemmas 3.7 and 10.22, the number of lattices $L_2 \oplus N$ in the sum in (11.5) is

$$\begin{aligned} & \frac{\text{Den}(A_{(1,(-1)^h}), A_{(1,1^h)}) / \text{Den}(A_{(1,(-1)^h}), A_{(1,(-1)^h)})}{\text{Den}(A_{((-1)^h)}, A_{(1^h)}) / \text{Den}(A_{((-1)^h)}, A_{((-1)^h)})} \\ &= \frac{\text{Den}(A_{(2,0^h)}, A_{(2,2^h)}) / \text{Den}(A_{(2,0^h)}, A_{(2,0^h)})}{\text{Den}(A_{(0^h)}, A_{(2^h)}) / \text{Den}(A_{(0^h)}, A_{(0^h)})} = q^{2h} \frac{1 - (-q)^{-h-1}}{1 - (-q)^{-1}}. \end{aligned}$$

This implies that

$$\begin{aligned} \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) &= q^{2h} \frac{1 - (-q)^{-h-1}}{1 - (-q)^{-1}} q^{-h-1}(q+1) \text{vol}(\langle x \rangle^{\vee})(1-q^{-2})^{-1} \\ &= q^h (1 - (-q)^{-h-1}) \text{vol}(\langle x \rangle^{\vee})(1-q^{-2})^{-1}. \end{aligned}$$

Now, by Theorem 10.19, Theorem 10.21, we have that

$$\widehat{\partial \text{Den}}_{L^{b\circ}, \mathcal{Y}}^{n,h}(x) = \widehat{\partial \text{Den}}_{L^{b\circ}}^{n,h}(x) - \widehat{\text{Int}}_{L^{b\circ}, \mathcal{H}}(x) = 0.$$

This proves (1) when $(a, b, c) = (0, h+1, n-h-2)$.

This finishes the proof of the theorem. \square

Proposition 11.3. (cf. [LZ22a, Proposition 7.3.4], [LL22, Proposition 2.22]) *Assume that L^b be an O_F -lattice of rank $n-1$ in \mathbb{V} . Then, $\partial \text{Den}_{L^b, \mathcal{Y}}^{n,h}(x)$ extends uniquely to a compactly supported locally constant function on \mathbb{V} (we still denote it by $\partial \text{Den}_{L^b, \mathcal{Y}}^{n,h}(x)$).*

Proof. Even though our situation is more similar to [LZ22a, Proposition 7.3.4], we do not have a functional equation like [LZ22a, (3.2.0.2)] yet. Therefore, let us follow the proof of [LL22, Proposition 2.22]. Basically, the proof is very similar to the proof of Theorem 11.2 (1).

Note that if L^b is not integral, then $\partial \text{Den}_{L^b, \mathcal{Y}}^{n,h}(x) = 0$. Therefore, we can assume that L^b is integral. Now, it suffices to show that for every $y \in L_F^b / L^b$, there exists an integer $\delta(y) > 0$

such that $\partial \text{Den}_{L^b, \gamma}^{n,h}(y+x)$ is constant for $x \in \pi^{\delta(y)}((L_F^b)^\perp)^{\geq 0} \setminus \{0\}$, where $\mathbb{V} = L_F^b \oplus (L_F^b)^\perp$ and $((L_F^b)^\perp)^{\geq 0} = \{(L_F^b)^\perp \mid (x, x) \in O_F\}$. If $L^b + \langle y \rangle$ is not integral, then there exists $\delta(y)$ such that $L^b + \langle y+x \rangle$ is not integral for $x \in \pi^{\delta(y)}((L_F^b)^\perp)^{\geq 0} \setminus \{0\}$, and hence $\partial \text{Den}_{L^b, \gamma}^{n,h}(y+x) = 0$. Therefore, we can assume that $L^b + \langle y \rangle$ is integral. Let $(a_1, a_2, \dots, a_{n-1})$ ($a_1 \leq a_2 \leq \dots \leq a_{n-1}$) be the fundamental invariants of L^b and let $\delta(y) = a_{n-1} + 2$. Then, it suffices to show that for a fixed pair (f_1, f_2) of generators of $((L_F^b)^\perp)^{\geq 0}$, we have

$$\partial \text{Den}_{L^b, \gamma}^{n,h}(y + \pi^\delta f_1) - \partial \text{Den}_{L^b, \gamma}^{n,h}(y + \pi^{\delta-1} f_2) = 0,$$

for $\delta > \delta(y) = a_{n-1} + 2$.

Now, let us introduce some notations from [LL22, Lemma 2.24]. Let \mathfrak{L} be the set of O_F -lattices in \mathbb{V} containing L^b , and let \mathfrak{C} be the set of triples (L^b, δ, ϵ) such that L^b is an O_F -lattice of L_F^b containing L^b , $\delta \in \mathbb{Z}$, and $\epsilon : \pi^\delta((L_F^b)^\perp)^{\geq 0} \rightarrow L^b \otimes_{O_F} F/O_F$ is an O_F -linear map. Then, consider the map $\theta : \mathfrak{L} \rightarrow \mathfrak{C}$ sending L to $(L \cap L_F^b, \delta_L, \epsilon_L)$ where δ_L is the maximal integer such that the image of L under the projection $\text{Pr}_\perp : \mathbb{V} \rightarrow (L_F^b)^\perp$ is contained in $\pi^{\delta_L}((L_F^b)^\perp)^{\geq 0}$, and ϵ_L is the extension map $\pi^{\delta_L}((L_F^b)^\perp)^{\geq 0} \rightarrow L^b \otimes_{O_F} F/O_F$ induced by the short exact sequence

$$0 \rightarrow L \cap L_F^b \rightarrow L \rightarrow \pi^{\delta_L}((L_F^b)^\perp)^{\geq 0} \rightarrow 0.$$

Then, as in the proof of [LL22, Lemma 2.24 (1)], θ is a bijection and its inverse is given by sending (L^b, δ, ϵ) to the O_F -lattice L generated by L^b and $\epsilon(x) + x$ for every $x \in \pi^\delta((L_F^b)^\perp)^{\geq 0}$.

Now, for every $\delta' \in \mathbb{Z}$, we define the following sets

$$\begin{aligned} \mathfrak{L}_1^{\delta'} &:= \{L \in \mathfrak{L} \mid L \subset L^\vee, \delta_L = \delta', y + \pi^\delta f_1 \in L\}, \\ \mathfrak{L}_2^{\delta'} &:= \{L \in \mathfrak{L} \mid L \subset L^\vee, \delta_L = \delta', y + \pi^{\delta-1} f_2 \in L\}, \end{aligned}$$

and for $L^b \subset L^b$, we define

$$\begin{aligned} \mathfrak{L}_{1, L^b}^{\delta'} &:= \{L \in \mathfrak{L} \mid L \subset L^\vee, \delta_L = \delta', y + \pi^\delta f_1 \in L, L \cap L_F^b = L^b\}, \\ \mathfrak{L}_{2, L^b}^{\delta'} &:= \{L \in \mathfrak{L} \mid L \subset L^\vee, \delta_L = \delta', y + \pi^{\delta-1} f_2 \in L, L \cap L_F^b = L^b\}. \end{aligned}$$

Since $\partial \text{Den}_{L^b, \gamma}^{n,h}(x)$ is a certain sum of $\partial \text{Den}_{L^b, \gamma}^{n,h}(x)$, it suffices to show that

$$\partial \text{Den}_{L^b, \gamma}^{n,h}(y + \pi^\delta f_1) - \partial \text{Den}_{L^b, \gamma}^{n,h}(y + \pi^{\delta-1} f_2) = 0,$$

for $\delta > \delta(y) = a_{n-1} + 2$ and $L^b \subset L^b$.

By definition, we have that

$$\begin{aligned} \partial \text{Den}_{L^b, \gamma}^{n,h}(y + \pi^\delta f_1) &= \sum_{\delta' \leq \delta} \sum_{L \in \mathfrak{L}_{1, L^b}^{\delta'}} D_{n,h}(L), \\ \partial \text{Den}_{L^b, \gamma}^{n,h}(y + \pi^{\delta-1} f_2) &= \sum_{\delta' \leq \delta-1} \sum_{L \in \mathfrak{L}_{2, L^b}^{\delta'}} D_{n,h}(L). \end{aligned}$$

Therefore, it suffices to show that

$$(11.6) \quad \sum_{\delta' \leq \delta} \sum_{L \in \mathfrak{L}_{1, L^b}^{\delta'}} D_{n,h}(L) - \sum_{\delta' \leq \delta-1} \sum_{L \in \mathfrak{L}_{2, L^b}^{\delta'}} D_{n,h}(L) = 0,$$

for all $\delta > \delta(y) = a_{n-1} + 2$ and $L^b \subset L^b$.

Since $\delta > a_{n-1} + 2$, we have that for $\delta' \leq 2$, we have

$$\mathfrak{L}_{1,L^b}^{\delta'} = \mathfrak{L}_{2,L^b}^{\delta'} = \{L \in \mathfrak{L} \mid L \subset L^\vee, \delta_L = \delta', y \in L, L \cap L_F^b = L^b\}.$$

Therefore, (11.6) equals to

$$\sum_{\delta'=2}^{\delta} \sum_{L \in \mathfrak{L}_{1,L^b}^{\delta'}} D_{n,h}(L) - \sum_{\delta'=2}^{\delta-1} \sum_{L \in \mathfrak{L}_{2,L^b}^{\delta'}} D_{n,h}(L) = 0.$$

Also, the automorphism of \mathfrak{C} sending (L^b, δ', ϵ) to $(L^b, \delta' - 1, \epsilon(\pi\alpha \cdot))$, where $\alpha \in O_F^\times$, $f_1 = \alpha f_2$, induces a bijection from $\mathfrak{L}_{1,L^b}^{\delta'}$ to $\mathfrak{L}_{2,L^b}^{\delta'-1}$. Therefore, it suffices to show that

$$(11.7) \quad \sum_{L \in \mathfrak{L}_{1,L^b}^2} D_{n,h}(L) = 1_{L^b}(y) \quad \sum_{L \subset L^\vee, L \cap L_F^b = L^b, \delta_L = 2} D_{n,h}(L) = 0.$$

Note that by Theorem 11.2 (1), we have that for $\text{val}((x, x)) \leq -2$,

$$\begin{aligned} \widehat{\partial \text{Den}}_{L^{b \circ}}^{n,h}(x) &= \sum_{L^b \subset L \subset L^\vee, L \cap L_F^b = L^b, x \in L^\vee} D_{n,h}(L) \text{vol}(L) \\ &= \text{vol}(L^b) \sum_{L^b \subset L \subset L^\vee, L \cap L_F^b = L^b, x \in L^\vee} D_{n,h}(L) \text{vol}(\text{Pr}_\perp(L)) \\ &= 0. \end{aligned}$$

Now, choose x to be generators of $\pi^{-2}((L_F^b)^\perp)^{\geq 0}$ and $\pi^{-3}((L_F^b)^\perp)^{\geq 0}$ and then take the difference. Then, we have that

$$\sum_{L \subset L^\vee, L \cap L_F^b = L^b, \delta_L = 2} D_{n,h}(L) = 0.$$

This shows that (11.7) holds which finishes the proof of the proposition. \square

Theorem 11.4. *Assume that Conjecture 6.3 above holds for $\mathcal{N}_n^{[h]}$ and Conjecture 7.6 holds for \mathcal{Z} -cycles in $\mathcal{N}_{n-1}^{[h-1]}$. Then, Conjecture 7.6 holds for \mathcal{Z} -cycles in $\mathcal{N}_n^{[h]}$.*

Proof. As in [LZ22a, section 8.2], we will prove this inductively. Let $L^b \subset \mathbb{V}$ be a rank $n - 1$ lattice such that L_F^b is non-degenerate, and let $x \in \mathbb{V} \setminus L_F^b$. By definition of $\partial \text{Den}_{L^b, \gamma}^{n,h}(x)$ and $\partial \text{Den}_{L^{b \circ}, \gamma}^{n,h}(x)$, it suffices to show that

$$(11.8) \quad \partial \text{Den}_{L^b, \gamma}^{n,h}(x) = \text{Int}_{L^b, \gamma}(x),$$

or equivalently

$$(11.9) \quad \partial \text{Den}_{L^{b \circ}, \gamma}^{n,h}(x) = \text{Int}_{L^{b \circ}, \gamma}(x),$$

for all $L^b \subset L^b \subset L^{b \vee}$.

Now, assume that (11.8) holds for L^b such that $\text{val}(L^b) < \text{val}(L^b)$.

Let $(a_1, a_2, \dots, a_{n-1})$ ($0 \leq a_1 \leq \dots \leq a_{n-1}$) be the fundamental invariants of L^b . Let $M = M(L^b) = L^b \oplus \langle u \rangle$ for some $u \in \mathbb{V}$ such that $\text{val}((u, u)) = a_n := a_{n-1}$ or $a_{n-1} + 1$, so that

$$a_1 + a_2 + \dots + a_n \equiv h + 1 \pmod{2}.$$

Now, assume that $(a'_1, a'_2, \dots, a'_n)$ be the fundamental invariants of the lattice $L^b + \langle x \rangle$ with a basis (e'_1, \dots, e'_n) such that $\text{val}(e'_i, e'_i) = a'_i$. Let $L'^b = \langle e'_1, \dots, e'_{n-1} \rangle$ and let $x' = e'_n$. Then, we have that

$$\text{Int}_{L^b}(x) = \text{Int}_{L'^b}(x), \quad \partial \text{Den}_{L^b}^{n,h}(x) = \partial \text{Den}_{L'^b}^{n,h}(x).$$

By [LZ22a, Lemma 8.2.2], if $x \notin M$, then $\text{val}(L'^b) < \text{val}(L^b)$, and hence by the inductive hypothesis, we have that

$$\text{Int}_{L'^b}(x) = \partial \text{Den}_{L'^b}^{n,h}(x).$$

This implies that the support of

$$\phi = \text{Int}_{L^b, \mathcal{Y}}(x) - \partial \text{Den}_{L^b, \mathcal{Y}}^{n,h}(x) \in C_c^\infty(\mathbb{V})$$

is contained in the lattice M . Here, $\text{Int}_{L^b, \mathcal{Y}}(x) - \partial \text{Den}_{L^b, \mathcal{Y}}^{n,h}(x)$ is in $C_c^\infty(\mathbb{V})$ by Proposition 11.3, Theorem 6.5, [LZ22a, Lemma 6.2.1], and [San17, Lemma 2.11].

Now, let us consider x such that $\text{val}((x, x)) < 0$ and $x \perp L^b$. Since we have assumed that Conjecture 6.3 below holds, by [LZ22a, Lemma 6.3.1] and [Zha22, Theorem 8.1], we have that

$$\chi(\mathcal{N}_n^{[h]}, \widehat{\mathbb{L}\mathcal{Z}(L^b)}_{\mathcal{Y}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) = \widehat{\text{Int}}_{L^b, \mathcal{Y}}(x) = -\frac{1}{q^h} \text{Int}_{L^b, \mathcal{Y}}(\mathcal{Y}(x)) := -\frac{1}{q^h} \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{Y}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}).$$

From now on, we use $\text{Int}_{L^b, \mathcal{Y}}(\mathcal{Y}(x))$ (resp. $\text{Int}_{L^{b\circ}, \mathcal{Y}}(\mathcal{Y}(x))$) to denote $\chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{Y}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)})$ (resp. $\chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{Y}}^{\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)})$).

Furthermore, note that we can decompose this into primitive parts

$$\begin{aligned} -\frac{1}{q^h} \text{Int}_{L^b, \mathcal{Y}}(\mathcal{Y}(x)) &= \sum_{L^b \subset L'^b \subset L'^{b\vee}} -\frac{1}{q^h} \text{Int}_{L'^{b\circ}, \mathcal{Y}}(\mathcal{Y}(x)) \\ &= \sum_{L^b \subset L'^b \subset L'^{b\vee}} \widehat{\text{Int}}_{L'^{b\circ}, \mathcal{Y}}(x). \end{aligned}$$

Now, let us compare this with the analytic side.

When $\text{val}((x, x)) \leq -2$, we have that $\mathcal{Y}(x)$ is empty. Therefore, combining this with Theorem 11.2, we have that

$$\widehat{\phi}(x) = \widehat{\text{Int}}_{L^b, \mathcal{Y}}(x) - \widehat{\partial \text{Den}}_{L^b, \mathcal{Y}}^{n,h}(x) = 0 - 0 = 0.$$

When $\text{val}((x, x)) = -1$, by Proposition 2.6, we have that

$$\begin{aligned} \widehat{\text{Int}}_{L^{b\circ}, \mathcal{Y}}(x) &= -\frac{1}{q^h} \text{Int}_{L^{b\circ}, \mathcal{Y}}(\mathcal{Y}(x)) = -\frac{1}{q^h} \chi(\mathcal{N}_n^{[h]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{Y}}^{\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}) \\ &= -\frac{1}{q^h} \chi(\mathcal{N}_{n-1}^{[h-1]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{Y}}^{\circ}) \\ &= -\frac{1}{q^h} \chi(\mathcal{N}_{n-1}^{[h-1]}, \mathbb{L}\mathcal{Z}(L^b)^{\circ} - \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}^{\circ}). \end{aligned}$$

Since we have assumed that Conjecture 7.6 holds for $\mathcal{N}_{n-1}^{[h-1]}$, we have that

$$-\frac{1}{q^h} \chi(\mathcal{N}_{n-1}^{[h-1]}, \mathbb{L}\mathcal{Z}(L^b)^{\circ}) = -\frac{1}{q^h} D_{n-1, h-1}(a, b, c),$$

where $(a, b, c) = (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$ and $\lambda \in \mathcal{R}_{n-1}^{0+}$ is the fundamental invariants of L^b .

For $\chi(\mathcal{N}_{n-1}^{[h-1]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}^{\circ})$, we have the following two cases.

First, assume that $(a, b, c) \neq (1, h-2, n-h)$. Then, by Theorem 5.3 (or see the proof of Theorem 11.2), we have that $\mathcal{Z}(L^b)_{\mathcal{H}}^\circ$ is empty or the sum of $\mathcal{Z}(L)^\circ \cdot \mathcal{Y}(N)$ for some lattice $N \simeq (\pi^{-1})^h$. Since $\mathcal{Y}(N)$ is empty in $\mathcal{N}_{n-1}^{[h-1]}$, we have that $\chi(\mathcal{N}_{n-1}^{[h-1]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}^\circ) = 0$.

Now, assume that $(a, b, c) = (1, h-2, n-h)$. Then by Theorem 5.3 (or see the proof of Theorem 11.2), we have that

$$\mathcal{Z}(L^b)_{\mathcal{H}}^\circ = \mathcal{Z}(L_2)^\circ \cdot \mathcal{Y}(\pi^{-1}L_1) \cdot \mathcal{Z}(L_0),$$

where $L_2 \simeq \pi^\lambda$ ($\lambda \geq 2$), $\pi^{-1}L_1 \simeq \pi^{-1}I_{h-2}$, and $L_0 \simeq I_{n-h}$. Therefore,

$$\chi(\mathcal{N}_{n-1}^{[h-1]}, \mathbb{L}\mathcal{Z}(L^b)_{\mathcal{H}}^\circ) = \chi(\mathcal{N}_1^{[1]}, \mathbb{L}\mathcal{Z}(L_2)^\circ) = D_{1,1}(1, 0, 0) = 1.$$

Combining these, we have that

$$\widehat{\text{Int}}_{L^b, \gamma}(x) = \begin{cases} -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) & \text{if } (a, b, c) \neq (1, h-2, n-h), \\ -\frac{1}{q^h} D_{n-1, h-1}(a, b, c) + \frac{1}{q^h} & \text{if } (a, b, c) = (1, h-2, n-h). \end{cases}$$

Therefore, by Theorem 11.2, we have that

$$\widehat{\text{Int}}_{L^b, \gamma}(x) - \widehat{\partial \text{Den}}_{L^b, \gamma}^{n, h}(x) = 0 \text{ for } \text{val}((x, x)) < 0.$$

This implies that $\widehat{\phi} = 0$ for $\text{val}((x, x)) < 0$.

Now, we only need to follow the proof of [LZ22a, Theorem 8.2.1].

Since ϕ is invariant under L^b and $\text{Supp}(\phi) \subset M$, we have that

$$\phi = 1_{L^b} \otimes \phi_\perp,$$

where $\phi_\perp \in C_c^\infty((L_F^b)^\perp)$ and ϕ_\perp is supported on $M_\perp = \langle u \rangle$. Then, we have

$$\widehat{\phi} = \text{vol}(L^b) 1_{L^b \vee} \otimes \widehat{\phi}_\perp,$$

and $\widehat{\phi}_\perp$ is invariant under the translation by $\langle u^\vee \rangle = \langle \pi^{-a_n} u \rangle$. Since $\text{val}((u^\vee, u^\vee)) = -a_n < 0$ and $\widehat{\phi}_\perp = 0$ for every x such that $x \perp L^b$, $\text{val}((x, x)) < 0$, we have that $\widehat{\phi}_\perp$ vanishes identically and hence $\phi = 0$. This finishes the proof of the Theorem. \square

Theorem 11.5. (cf. [LZ22a, Theorem 8.2.1, Theorem 10.5.1]) *Conjecture 7.6 holds for \mathcal{Z} -cycles in $\mathcal{N}_n^{[0]}, \mathcal{N}_n^{[1]}, \mathcal{N}_n^{[n-1]}, \mathcal{N}_n^{[n]}$, and $\mathcal{N}_4^{[2]}$.*

Proof. This follows from Theorem 11.4 and Theorem 6.5. \square

Remark 11.6. We remark that the Kudla-Rapoport conjecture for $\mathcal{N}_n^{[0]}, \mathcal{N}_n^{[n]}$ (good reductions case), $\mathcal{N}_n^{[1]}$ (almost self-dual case) is already proved in [LZ22a]. Therefore, our new cases are $\mathcal{N}_n^{[n-1]}$ and $\mathcal{N}_4^{[2]}$. We also remark that our work gives a different proof of the conjecture for $\mathcal{N}_n^{[1]}$.

REFERENCES

- [BKP⁺18] S. C. Billey, M. Konvalinka, T. K. Petersen, W. Slofstra, and B. E. Tenner, *Parabolic double cosets in Coxeter groups*, Electron. J. Combin. **25** (2018), no. 1, Paper No. 1.23, 66. MR3785002 [↑25](#)
- [BL00] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics, vol. 182, Birkhäuser Boston, Inc., Boston, MA, 2000. MR1782635 [↑25](#)
- [Cho18] S. Cho, *The basic locus of the unitary Shimura variety with parahoric level structure, and special cycles*, 2018. [↑10, 11, 24, 25, 26, 27](#)

- [Cho22a] ———, *On local representation densities of hermitian forms and special cycles II*, 2022. [↑17](#), [81](#), [82](#)
- [Cho22b] ———, *Special cycles on unitary Shimura varieties with minuscule parahoric level structure*, *Mathematische Annalen* (2022), 1–67. [↑2](#), [5](#), [28](#), [29](#), [30](#), [31](#), [32](#), [33](#), [35](#)
- [Cho23] ———, *On Local Representation Densities of Hermitian Forms and Special Cycles*, *International Mathematics Research Notices* (2023). [↑7](#), [8](#), [30](#), [31](#), [32](#), [35](#), [36](#), [37](#), [38](#), [39](#), [40](#), [83](#)
- [CY20] S. Cho and T. Yamauchi, *A reformulation of the Siegel series and intersection numbers*, *Mathematische Annalen* **377** (2020), no. 3, 1757–1826. [↑15](#), [32](#), [33](#)
- [DL22] D. Disegni and Y. Liu, *A p -adic arithmetic inner product formula*, arXiv e-prints (Apr. 2022), arXiv:2204.09239, available at [2204.09239](#). [↑2](#)
- [GHN24] U. Görtz, X. He, and S. Nie, *Basic loci of coxeter type with arbitrary parahoric level*, *Canadian Journal of Mathematics* **76** (2024), no. 1, 126–172. [↑25](#)
- [Hir00] Y. Hironaka, *Classification of hermitian forms by the iwahori subgroup and their local densities*, *Commentarii Mathematici Universitatis Sancti Pauli* (2000). [↑29](#)
- [HLSY23] Q. He, C. Li, Y. Shi, and T. Yang, *A proof of the Kudla–Rapoport conjecture for Kramer models*, *Inventiones mathematicae* **234** (2023), 721–817. [↑2](#), [4](#), [5](#), [6](#), [7](#), [8](#), [14](#), [61](#), [62](#)
- [How19] B. Howard, *Linear invariance of intersections on unitary rapoport–zink spaces*, *Forum mathematicum*, 2019, pp. 1265–1281. [↑12](#), [13](#)
- [HSY20] Q. He, Y. Shi, and T. Yang, *The Kudla–Rapoport conjecture at a ramified prime for $U(1,1)$* , *Transactions of the AMS*, to appear. (2020). [↑2](#)
- [HSY23] ———, *Kudla–Rapoport conjecture for Kramer models*, *Compositio Mathematica* **159** (2023), no. 8, 1673–1740. [↑2](#), [4](#), [14](#), [15](#), [20](#)
- [KR00] S. Kudla and M. Rapoport, *Height pairings on Shimura curves and p -adic uniformization*, *Invent. Math.* **142** (2000), no. 1, 153–223. [MR1784798](#) [↑2](#)
- [KR11] ———, *Special cycles on unitary Shimura varieties I. Unramified local theory*, *Invent. Math.* **184** (2011), no. 3, 629–682. [MR2800697](#) [↑2](#), [5](#), [10](#), [22](#), [23](#), [31](#), [32](#), [85](#), [86](#), [87](#), [88](#)
- [KR14] ———, *Special cycles on unitary Shimura varieties II: Global theory*, *J. Reine Angew. Math.* **697** (2014), 91–157. [MR3281653](#) [↑11](#)
- [Kud04] S. Kudla, *Special cycles and derivatives of Eisenstein series*, *Heegner points and Rankin L -series*, 2004, pp. 243–270. [MR2083214](#) [↑1](#)
- [Kud97] ———, *Central derivatives of eisenstein series and height pairings*, *Annals of mathematics* **146** (1997), no. 3, 545–646. [↑1](#)
- [LL21] C. Li and Y. Liu, *Chow groups and L -derivatives of automorphic motives for unitary groups*, *Ann. of Math. (2)* **194** (2021), no. 3, 817–901. [MR4334978](#) [↑2](#)
- [LL22] ———, *Chow groups and L -derivatives of automorphic motives for unitary groups, II.*, *Forum of mathematics*, pi, 2022. [↑2](#), [5](#), [7](#), [88](#), [89](#)
- [LS90] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $Sl(n)/B$* , *Proc. Indian Acad. Sci. Math. Sci.* **100** (1990), no. 1, 45–52. [MR1051089](#) [↑25](#)
- [Lus76] G. Lusztig, *Coxeter orbits and eigenspaces of frobenius*, *Inventiones mathematicae* **38** (1976), no. 2, 101–159. [↑26](#), [27](#)
- [LZ22a] C. Li and W. Zhang, *Kudla–Rapoport cycles and derivatives of local densities*, *Journal of the American Mathematical Society* **35** (2022), no. 3, 705–797. [↑2](#), [5](#), [6](#), [7](#), [11](#), [12](#), [15](#), [16](#), [19](#), [23](#), [24](#), [26](#), [27](#), [33](#), [56](#), [85](#), [88](#), [90](#), [91](#), [92](#)
- [LZ22b] ———, *On the arithmetic Siegel–Weil formula for $GSpin$ Shimura varieties*, *Inventiones mathematicae* **228** (2022), no. 3, 1353–1460. [↑5](#), [7](#), [8](#), [61](#), [62](#)
- [Ohm10] J. Ohmori, *The tate conjecture over finite fields for projective schemes related to coxeter orbits*, *Hokkaido Mathematical Journal* **38** (2010), 1–38. [↑26](#), [27](#)
- [RZ96] M. Rapoport and T. Zink, *Period spaces for p -divisible groups*, Princeton University Press, 1996. [↑10](#)

- [San17] S. Sankaran, *Improper intersections of kudla–rapoport divisors and eisenstein series*, Journal of the Institute of Mathematics of Jussieu **16** (2017), no. 5, 899–945. [↑2](#), [6](#), [91](#)
- [Sie35] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. (2) **36** (1935), no. 3, 527–606. [MR1503238](#) [↑1](#)
- [Sie51] ———, *Indefinite quadratische Formen und Funktionentheorie. I*, Math. Ann. **124** (1951), 17–54. [MR0067930](#) [↑1](#)
- [Vol10] I. Vollaard, *The supersingular locus of the shimura variety for $gu(1, s)$* , Can. J. Math. **62** (2010), 668–720. [↑25](#)
- [VW11] I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety of $GU(1, n-1)$ II*, Inventiones mathematicae **184** (2011), no. 3, 591–627. [↑65](#)
- [Wei65] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87. [MR0223373](#) [↑1](#)
- [Zha22] Z. Zhang, *Maximal parahoric arithmetic transfers, resolutions and modularity*, 2022. [↑6](#), [19](#), [24](#), [86](#), [87](#), [88](#), [91](#)

DEPARTMENT OF MATHEMATICS, POSTECH, 77, CHEONGAM-RO, NAM-GU, POHANG-SI, GYEONGSANGBUK-DO, KOREA

Email address: sungyooncho@postech.ac.kr

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027, USA

Email address: qh2275@columbia.edu

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, 450 JANE STANFORD WAY BUILDING 380, STANFORD, CA 94305

Email address: zyuzhang@stanford.edu